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QUANTIFIER ELIMINATION IN APPLIED MECHANICS PROBLEMS WITH CYLINDRICAL ALGEBRAIC DECOMPOSITION

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Abstract—The method of cylindrical algebraic decomposition (CAD) of the k-dimensional space constitutes a classical technique for the efficient solution of quantifier elimination (QE) problems in algorithmic, computer-aided algebra. Here we apply this method to some applied mechanics problems under appropriate constraints. At first, we study the problem of a straight elastic beam under a restriction on the maximum permissible deflection along this beam (which can easily be reduced to the construction of a one-dimensional CAD) as well as the problem of a circular isotropic elastic medium where a stress component should not exceed a critical value (which requires the construction of a three-dimensional CAD). In both these problems, we derive also the required quantifier-free formulae (QFFs) not including the fundamental variables, but only the parameters involved. Much more difficult CAD/QFF-derivation applications, concerning an elliptical elastic medium again with an upper bound for a stress component, a special case of failure by yielding in fracture mechanics, related to Sih's strain-energy-density factor, and a frictionless contact problem for an elastic half-plane are also considered and explicitly solved with the help of already available CAD-produced results although, evidently, CAD is not expected to produce QFFs in extremely difficult problems. Finally, additional possible applications of CAD/CQE to applied mechanics problems are also suggested. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Computer algebra systems (CASs) such as Axiom (Jenks and Sutor, 1992; NAG, 1996), Macsyma (Computer Aided Mathematics Group of Symbolics, Inc., 1988), Maple V (Char *et al.*, 1991; Heal *et al.*, 1996; Monagan *et al.*, 1996; Corless, 1995), Mathematica (Wolfram, 1996; Maeder, 1996), Reduce (Brackx and Constales, 1991) and SACLIB (Buchberger *et al.*, 1993) have been used for the solution of a large number of problems in applied mechanics including symbolic, numerical and mixed computations. A recent, detailed comparison of seven popular CASs was prepared by Wester (1995). The well-known book by Davenport *et al.* (1993) is a standard reference on computer algebra algorithms. Among a variety of related applied mechanics applications, we can mention the case where we have problems involving a parameter (or more than one parameter). Several such problems in applied mechanics were recently considered and solved by using the symbolic/numerical capabilities of computer algebra systems (see, e.g., Ioakimidis (1993), Ioakimidis and Anastasselos (1992)), which permit computing simultaneously in the numerical and the symbolic environments.

In the above references, general-purpose computer algebra commands were employed for the solution of the problems under consideration. The next step was to use rather special computer algebra algorithms (Davenport *et al.*, 1993; Mishra, 1993) as a tool in our applications. Such a fundamental algorithm is the famous Buchberger algorithm for the construction of Gröbner bases in systems of polynomial equations (Davenport *et al.*, 1993; Mishra, 1993). This algorithm was recently used together with Maple V in a variety of applied mechanics and elasticity applications (see, e.g., Ioakimidis and Anastasselou (1993)). (A list of the related references is reported by Ioakimidis (1996a).) The somewhat related algorithm for the construction of characteristic sets (Mishra, 1993; Wang, 1995) was recently used (Anastasselou, 1995) in the same class of problems. It is this author's personal opinion that these algorithms can prove very efficient for the solution of applied mechanics problems and they offer new possibilities in the related research.

One further step in the use of computer algebra algorithms in applied mechanics is offered by Sturm's sequences and the classical Sturm theorem in algorithmic, computer-aided algebra (Davenport *et al.*, 1993; Mishra, 1993). The use of this theorem permitted us to arrive at decisions in simple elasticity problems including quantifiers (Ioakimidis, 1996a), mainly the universal quantifier $\forall$ (for all) although the existential quantifier $\exists$ (exists) can also be used, but no parameters. The examples in this paper concerned: (i) a simple beam problem (decide whether its deflection, under a known loading, reaches a similarly known critical value or not), (ii) a straight crack problem (decide whether, under a known pressure distribution on the crack edges, these edges come into contact or not), (iii) a simple plane elasticity problem (decide whether the stress component σ_y reaches a known critical value σ_0 inside the circular region assumed occupied by the isotropic elastic medium or not) and, finally, (iv) a caustic problem (decide whether the caustic formed around a crack tip in fracture mechanics lies inside a known circular region or not). These simple applied mechanics problems were used by Ioakimidis (1996a) as the vehicle for the illustration of the usefulness and powerfulness of computer algebra systems for the solution of simple decision problems.

The next obvious possibility was to consider parameters in the decision problems involving quantifiers. Then the decision will not be simply `true` or `false`, but it will generally depend on the particular values of the parameters involved. In such a case, we speak about quantifier elimination (QE) problems, better about computational quantifier elimination (CQE) problems and the solution to such a problem will be a formula (possibly consisting of several atomic formulae combined by the "and" and "or" logical operators, conjunctions and disjunctions, respectively) not including the quantifiers and the quantified variables, but only the parameters involved. This possibility has been recently studied by the author, who used three related elementary approaches. More explicitly, the following approaches have been employed: (i) the Descartes rule of signs (Ioakimidis, 1995a, 1996b, d, 1997a) kindly suggested by Collins in a private communication, who derived also the related results, (ii) classical Sturm sequences (Ioakimidis, 1996c) exactly as in simple decision problems (Ioakimidis, 1996a), and (iii) Sturm–Habicht sequences, a modification and improvement of the original Sturm sequences (Ioakimidis, 1995b, 1996b, e, 1997a) in a variety of applied mechanics problems.

In the present paper, our aim is to employ a much more general and systematic algorithm for the solution of applied mechanics problems. The algorithm that we will use here is the classical cylindrical algebraic decomposition algorithm (simply CAD among the experts, not to be confused with computer-aided design), which will be briefly described in the next section. This algorithm was originally suggested by Collins in 1973 and considerably improved by Collins and his collaborators since that year. On the other hand, CAD is one of the very few existing general-purpose CQE algorithms not restricted to special CQE problems contrary to several other CQE algorithms such as the elementary, special-purpose algorithm for linear inequalities having been used by Ioakimidis (1997b) in a fracture mechanics problem or the much better known Fourier elimination (Williams, 1986). Moreover, CAD is the best computer-implemented CQE algorithm through the devoted `qepcad` SACLIB package. Under these circumstances, the use of CAD in general CQE problems in applied mechanics is strongly recommended.

After an introduction to the CAD method in Section 2, we will apply CAD to two rather elementary CQE problems in applied mechanics in Sections 3 and 4. These applications aim at the illustration of the CAD algorithm in practical CQE applied mechanics problems and by no means do they approach the actual limits of CAD neither have they been automatically derived by a computer package such as `qepcad`. In passing, we can mention that these problems constitute significant generalizations (through the introduction of parameters) of the first and the third aforementioned decision problems already having been studied by the author (Ioakimidis, 1996a). Few related computational details are reported in Section 5. Next, in Section 6, we will show the applicability of CAD to three much more difficult and, probably, interesting CQE applied mechanics problems: (i) a problem concerning an upper bound for a stress component in an elliptical elastic medium, (ii) a simple application of the classical Sih's strain-energy-density factor S to yielding about a crack tip, and (iii) the similarly classical frictionless punch problem for an elastic half-plane concerning the complete contact between the punch and the half-plane. In the results of this section, we used already available CAD-derived quantifier-free formulae (QFFs). The existing practical limitations of CAD become clear from the CQE problems of Section 7. Next, some further CQE problems in applied mechanics which may be solved by CAD or by competitive CQE algorithms are reported in Section 8. Finally, in Section 9, some further comments on CAD are made and our present conclusions are drawn.

2. ON THE CAD ALGORITHM

2.1. *The principles of the CAD algorithm*

As far as the cylindrical algebraic decomposition (CAD) is concerned, it is an algorithmic method in computer-aided algebra which permits us to decompose our space of k real variables, including both the ordinary variables, such as x, y and z for the Cartesian coordinates and t for time, and the parameters (auxiliary or free variables) appearing in our applications such as the geometry, material, loading and strength/failure parameters in applied mechanics. This decomposition, which is algebraic, cylindrical and cellular in CAD, permits the partition of our generalized space into a finite number of appropriate cylindrically arranged disjoint, simply-connected semi-algebraic sets (defined by Boolean combinations of finite sets of polynomial equations and/or inequalities with algebraic coefficients), called cells and, frequently, mainly for computational purposes, appropriately further combined into clusters, where we can always use just one concrete and essentially arbitrarily-selected sample point for reaching our conclusion for the whole cell or, better, cluster.

More explicitly, we assume that we have a real quantified formula (QF) Q in elementary algebra and geometry (EAG) consisting of one or more atomic (elementary) formulae including only an easily extractable finite set A of n real, different, r-variate ($r \geqslant 1$) polynomials A_i ($i = 1, 2, \ldots, n$) with integer coefficients (with respect to some or all of the aforementioned variables and containing equality or inequality symbols, e.g. $A_i > 0$ or $A_i \leqslant 0$) and combined by the "and" and "or" Boolean–logical operators. Moreover, this (composite) formula (QF) Q is assumed, just for convenience, to be/have been put in standard prenex form, that is with the quantifiers ($\forall$ and $\exists$) appearing at the beginning of the formula. Among the r total variables, h are free variables (parameters) and $r-h$ are quantified variables and should be eliminated. What we look for is a related quantifier-free formula (QFF) Q^* completely equivalent to Q, but free both from the quantifiers involved at the beginning of Q as well as from the related $r-h$ (quantified) variables in the whole formula. This formula, the QFF Q^*, will contain only the h free variables–parameters or some of them or even none of them when they all yield just `true` or `false` in really exceptional cases.

A first step towards the above-described task is to construct a CAD of our generalized (r-dimensional, real) Euclidean space, including both the quantified and the free variables (parameters), consisting of cells (better clusters) as was mentioned above and with the property that all of our polynomials A_i entering into the QF Q are sign-invariant in all of the cells (or clusters) used. The construction of such a CAD consists of the following phases: (i) the projection phase, where appropriate successive sets of projection polynomials, B, C, ..., are computed (beginning with A in the r-dimensional space) with each such set having exactly one less variable in comparison with the previous set (and, normally, many more polynomials); finally, we can reach a finite set P of m univariate projection polynomials, e.g., in the free variable (parameter) u: $P_i(u)$ ($i = 1, 2, \ldots, m$), (ii) the base phase, where a CAD of the real u-axis is constructed and sample points are appropriately selected there, as will be explained in the next paragraph, and (iii) the extension (or "lifting") phase, where, on the basis of the cells of the u-axis and the selected sample points in them, the related one-dimensional CAD is successively extended to higher-dimensional Euclidean spaces: in two dimensions, (u, v), in three dimensions, (u, v, w), etc. up to the h-dimensional space of free variables. Generally, we do not have to proceed further, that is, up to the r-dimensional original space in Q. During this "lifting" phase, after the selection of l appropriate sample points in the u-axis, we essentially have only sets $R^{(j)}(j = 1, 2, \ldots, l)$ of "univariate" polynomials (with respect to v) in the two-dimensional space (u, v) (directly resulting from the set R of bivariate related polynomials $R_i(u, v)$ for the l "sample values" of u there) after the first "lifting" and, therefore, we can also select appropriate sample points there again just in one dimension (the v-axis) along the cylinders (in the ordinary meaning of this word) with bases the cells of the u-axis. In this way, we use the sections and sectors of these cylinders having been defined by the R-sets of polynomials.

Returning to the base phase, on the real u-axis only, we have the finite set P of m real integral polynomials $P_i(u)$ there and we wish to construct a P-sign-invariant CAD of the u-axis, that is a partition of the u-axis into a finite number of l disjoint cells C_j (either point cells, simple points, or open interval cells, both finite and infinite, directly defined by the point cells) having the property that all of the polynomials $P_i(u)$ keep the same sign (are sign-invariant): plus, minus or zero for zero values of a polynomial, in each separate cell C_j $(j = 1, 2, \ldots, l)$. This task is very easy: we have simply to compute all of the real zeros of all of the univariate polynomials $P_i(u)$ $(i = 1, 2, \ldots, m)$ say k such zeros, and use them as the point cells of the present univariate CAD. Obviously, $P_i(u)$ are sign-invariant at these trivial, point cells, but also in the $k+1$ open interval cells ($k-1$ finite and two infinite) defined by the k point cells, since any polynomial $P_i(u)$ may change its sign only at one of its zeros. Therefore, we finally have got $l = k+(k-1)+2 = 2k+1$ disjoint cells in the u-axis and we have thus constructed a CAD of this axis with respect to which all of the polynomials $P_i(u)$ are sign-invariant. This is the base phase of the CAD construction already reported in the previous paragraph, which, of course, has to be repeated with respect to all of the subsequent free variables: v, w, etc., but after the selection of the l concrete u-sample points as will be explained in the next paragraph. (We will have the opportunity to present two concrete applied mechanics applications of the CAD algorithms in Sections 3 and 4 below.)

Next, as far as the k real zeros (simple zeros; multiple zeros are considered also as simple ones) of the polynomials $P_i(u)$ are concerned, there are either rational numbers or, much more frequently, irrational algebraic numbers, which yet can be approximated to any desired precision. These numbers de facto constitute the point cells of the CAD of the u-axis. Quite frequently, we do not have explicit expressions for these numbers, but just sufficiently small open bounding intervals (limited by rational numbers) including them. Such intervals can be computed with the aid of a computer algebra system. In any case, next, it is easy to compute one sample point for each one of the $k+1$ finite and infinite open interval u-cells on the basis of the ends of the small bounding intervals of the aforementioned zeros. Then these sample points will be just rational numbers and this is strongly preferable. Therefore, finally, in the worst case, our $l = 2k+1$ sample points on the u-axis will include $k+1$ rational points and k irrational algebraic points. (To be honest, below we intend to use approximations to the irrational sample points instead of working with them since this is a very time-consuming task.) Now we are ready to proceed to the first "lifting" and work with each cylinder (in the v-direction) based on each particular cell of the u-axis. Along this cylinder we will use the related $R_i(u, v)$ polynomials (also with integral coefficients, a property inherited by the n original A_i polynomials in the QF Q), but just for the already selected $l = 2k+1$ u-sample points. Therefore, we will construct just l one-dimensional CADs again (but with respect to v now) along each such (two-dimensional) cylinder and so on for further "liftings" up to the h-dimensional space of the free variables. (A full set of CAD references will be mentioned in the next subsection.)

Of course, after having completed the CAD construction (with respect to the h, $h < r$, free variables–parameters only; this is sufficient), we can directly decide which are the cells (better clusters, appropriately selected unions of adjacent cells) where the original QF Q yields `true` and which are the cells where it yields `false`. Clearly, this will be done on the basis of the sample points only: one sample point in each cell (or cluster). In this almost final step, we have thus transformed our sign-invariant CAD of the real h-space into a *truth-invariant* CAD and, therefore, we have made a further step towards the construction of the QFF Q^* equivalent to Q. The final step for the construction of Q^* is to define a formula disjunctively collecting all of the cells (or clusters) where Q has already resulted to be `true` (and no cell/cluster where it yielded `false`). In principle, this can easily be achieved on the basis of the polynomial equations/inequalities defining each cell/cluster, but the resulting output may be extremely complicated. Therefore, very efficient simplification algorithms are normally required so that we can finally obtain a moderately long QFF Q^* although this is a rather difficult task requiring much expertise and it is not always feasible. In this way of thinking, CAD has been employed as a computational quantifier elimination (CQE) algorithm, but here for real variables in EAG (somewhat more generally, in real closed fields). At this point, a clear distinction from quantifier elimination (QE) in logic should be made, since most research papers on QE refer to logic, not to EAG.

Therefore, since CAD uses only one concrete sample point in each cell/cluster, it could be considered to be essentially a numerical method, but the situation is somewhat more complicated and the use of computer algebra systems (CASs) becomes more or less indispensable. At first, the projections in the first phase of CAD (from the r-dimensional space up to the one-dimensional u-space) are accomplished by employing computer algebra commands having to do with symbols such as those for the computation of the leading coefficient and the discriminant of a multivariate polynomial (with respect to its main variable) and the resultant of two such polynomials. Moreover, frequently, derivatives are also required as well as remainders in the division of two polynomials. The commands for Sturm's sequences and the number of distinct real zeros of a polynomial in a real interval may also be of interest.

Furthermore, as was already mentioned, quite frequently, we have to work with exact rational and irrational algebraic numbers (although this task will be avoided below through floating-point approximate computations). These computations, including exact arithmetic and sign determination algorithms for irrational algebraic numbers, cannot be performed inside the purely numerical environment offered by classical computer languages such as BASIC, FORTRAN and C and, therefore, the use of CASs is indispensable. Finally, the author also feels that the computational environment offered by CASs is often significantly superior to that offered by classical computer languages and, therefore, it can be preferred for medium-sized computations.

2.2. *CAD references*

Before proceeding to our applied mechanics applications, we wish to cite some publications concerning CAD. The conception of CAD is due to Collins (in 1973) as a nontrivial generalization of previous related (but much more complicated from the computational point of view) pioneering results by Tarski (1951) (originally obtained about 1930) as a continuation of his results in logic and the methodology of deduction (see, e.g., Tarski (1994)). The fundamental related full-length paper is that by Collins (1975). During the subsequent years the research team by Collins, Arnon, McCallum and, later, Hong investigated and improved the CAD method and proposed several algorithms for its faster performance in the computer environment. Few additional researchers offered also their significant contributions to CAD.

The most important of the related publications (beyond the original one by Collins (1975)) are those by Arnborg and Feng (1988), Arnon (1985a, b, 1988a–c), Arnon *et al.* (1984a, b, 1985, 1988), Arnon and McCallum (1982, 1985, 1988), Arnon and Mignotte (1988), Buchberger and Hong (1991), Collins (1975, 1976, 1982, 1983, 1995, 1997), Collins and Hong (1991), Collins and Johnson (1989), Collins and McCallum (1995), Davenport (1985), Davenport and Heintz (1988), Hong (1990a, 1991a, b, 1992, 1993c, d), Lazard (1994), McCallum (1985, 1988, 1993, 1997), Prill (1986), Richardson (1991, 1997) and Saunders *et al.* (1989). The Ph.D. theses by Arnon (1981), McCallum (1984) and Hong (1990b) (all three under the guidance of Collins) are also of particular interest today, especially the last one (Hong, 1990b). Moreover, Mishra (1993) devoted few sections of his book to CAD/CQE and, more recently, Winkler (1996) a whole short chapter.

Fortunately, the interested reader can easily consult four major sources of research results concerning CAD and CQE in general:

(i) A special issue of the *Journal of Symbolic Computation* on algorithms in real algebraic geometry, appeared also as a separate book (Arnon and Buchberger, 1988) and containing several CAD-related papers.

(ii) A special issue of *The Computer Journal* (Hong, 1993a) devoted to CQE in general and including several interesting papers such as those by McCallum (1993) and Liska and Steinberg (1993), the last of which is an application of CAD to the stability analysis of difference schemes for partial differential equations.

(iii) The significantly extended proceedings (Caviness and Johnson, 1997) of the 1993 CAD/CQE symposium on "Quantifier Elimination and Cylindrical Algebraic Decomposition" (most probably, the first one on CAD), having taken place at the Research Institute for Symbolic Computation of the Johannes Kepler University of Linz (RISC-Linz) in October 1993 and organized by Caviness and Buchberger. This symposium took place in honour of Collins (30 years after the invention of CAD). The proceedings of the symposium include not only recent CAD/CQE-related papers, mainly a very interesting review paper by Collins himself (1997), but also practically all of the main contributions to the CAD algorithms by Collins and his collaborators in chronological order of appearance (Collins, 1975; Arnon, Collins and McCallum, 1984a, b; Hong, 1990a; Collins and Hong, 1991, Hong, 1992; McCallum, 1997). These proceedings (Caviness and Johnson, 1997) were finally announced to appear in January 1997.

(iv) Finally, since 1995 special annual International IMACS Conferences on Applications of Computer Algebra (IMACS ACA Conferences) are being organized. The first such conference took place at the University of New Mexico, Albuquerque, in May 1995 and included a session on "Quantifier Elimination and Its Applications" (Wester *et al.*, 1995), organized by Liska and Jahn. Five interesting CQE papers were presented in this conference and two of them, those by Hong *et al.* (1995) and McCallum (1995), concerned CAD itself. Next, very recently, in July 1996, during the 2nd IMACS ACA Conference, now at RISC-Linz, a special session focusing on the theory and the applications of CQE also took place, organized by Steinberg, Hong and Liska. Among the interesting papers presented in this session, those by González-Campos and González-Vega (1996) and Brown and Collins (1996) concern theoretical aspects of the CAD algorithm, whereas the papers by Jirstrand and by Liska and Steinberg concern CQE applications in nonlinear control theory (Jirstrand, 1996) and the stability of boundary conditions in initial boundary value problems for partial differential equations and their finite-difference discretizations (Liska and Steinberg, 1996), where previous CAQ/CQE stability results by the same authors (Liska and Steinberg, 1993) are extended. The paper by Hong and Neubacher (1996), where approximate CQE is officially proposed, seems also to be of particular importance and test results to the stability and control theories are included in it. Unfortunately, the papers in this special CQE session have not appeared yet although, probably, this will take place during 1997 or 1998 in the *Journal of Symbolic Computation* and the IMACS Journal *Mathematics and Computers in Simulation*. The third IMACS ACA Conference will be held in Maui, Hawaii in July 1997 (just after ISSAC '97 and PASCO '97).

2.3. *CAD computer implementations, applications and CQE alternatives*

Of course, there are also special CAD packages incorporating the fundamental Collins' CAD algorithm and its various improvements. These are the packages prepared by Collins himself, Arnon, Hong and their collaborators. Originally, CAD was incorporated into the famous LISP-based SAC-2 computer algebra system and really used by Collins, Arnon, Hong and other researchers in the past for the completely mechanical (or appropriately human-intervened) construction of CADs and QFFs. More explicitly, the first complete CAD package has been prepared by Arnon, but Hong prepared a new and much more powerful CAD package, the famous qepcad package, including partial CAD (Collins and Hong, 1991) and several additional improvements by Hong (1990b). This package is now written in the C language, being a part of the SACLIB computer algebra system (Buchberger *et al.*, 1993), and it seems that it is the sole computer package on CAD which should be used today. Collins and Hong still actively work on the improvement of qepcad (which is already in a very advanced stage of development) and on the preparation of its manual.

Unfortunately, although SACLIB is freely distributed (through INTERNET) from RISC-Linz since 1993, this is not still the case with qepcad, which is intended to become a standard part of SACLIB. This is expected to take place soon, probably during 1997, and together with the final preparation of the related manual. Meanwhile, only very few researchers (McCallum, 1993; Liska and Steinberg, 1993; Jirstrand, 1996) have been able to efficiently use qepcad in practical CQE problems although by no means both Hong and Collins refuse its availability to the interested researcher (in spite of the present lack of the manual).

We believe that qepcad will be really helpful to the user of the CAD algorithm either in applied mechanics or in any other branch of applied mathematics, science and engineering. Of course, although the definitive public release of qepcad as a standard SACLIB package will be an extremely significant assistance to the CAD user, its adaptation so that it can be used together with a somewhat more popular computer algebra system (such as Maple V and Mathematica), even externally, should be encouraged and recommended.

Having been strongly interested in CAD in order to investigate the possibility of its application to applied mechanics for the solution of concrete CQE problems by a general-purpose, computer-based CQE algorithm, we found it reasonable to check whether this approach has been already adopted in applied mechanics in the past or not. Our related investigation has been completely negative as far as applied mechanics problems are concerned. On the contrary, this search revealed a rather old paper by Champetier and Magni (1988) in control theory as well as the aforementioned more recent papers by Liska and Steinberg (1993) in the stability analysis of difference schemes, the very recent paper by Jirstrand (1996) in nonlinear control theory and few additional mathematics-related papers. CAD is also well known to be of interest in motion planning problems in robotics (Marchand, 1989), in collision problems also in robotics (Hong, 1991a) and, of course, in the classical areas of automatic geometric theorem proving and curve analysis (Buchberger *et al.* 1988). Therefore, it seems that the CQE algorithm offered by CAD has not been still used in applied mechanics up to now (contrary to less general and efficient approaches already reported in the previous section) in spite of the fact that, in our opinion, it constitutes a really powerful tool for the solution of actual research and everyday problems.

Of course, beyond CAD alternative CQE algorithms, both general-purpose and special-purpose, are also available. Hong (1991b) compared three of these general-purpose CQE algorithms (CAD, the algorithm of Grigor'ev and Vorobjov and Renegar's algorithm) and he found that CAD is undoubtedly by far the best of them. Two more recent general-purpose CQE algorithms have been proposed by González-Vega (1993) and Weispfenning (1993), whereas among several special-purpose CQE algorithms we can make reference to those suggested by Loos and Weispfenning (1993) for some linear CQE problems and by Hong (1993b) for some quadratic CQE problems. As was already mentioned, approximate CQE was also recently officially proposed by Hong and Neubacher (1996).

3. A ONE-DIMENSIONAL ELASTICITY PROBLEM

At first, we consider the problem of the statically indeterminate propped cantilever straight elastic beam $[0, L]$ fixed at one end, $x = 0$, and simply-supported at the other end, $x = L$, under a constant loading q along the whole beam. If EI refers to the flexural rigidity of the beam (with E denoting the modulus of elasticity of the isotropic elastic material and I the appropriate moment of inertia of the cross-section of the beam), the deflection $v = v(x)$ of its points is given by (Timoshenko and Gere, 1973)

$$EIv = -\frac{qLx^3}{6}+\frac{R_bx^3}{6}+\frac{qL^2x^2}{4}-\frac{R_bLx^2}{2}+\frac{qx^4}{24}, \quad 0 \leqslant x \leqslant L, \tag{1}$$

where R_b denotes the reaction at the simply-supported end $x = L$ of the beam, easily determined to be equal to $3qL/8$. Then eqn (1) can be written in the simplified form:

$$EIv = \frac{qx^4}{24} - \frac{5qLx^3}{48} + \frac{qL^2x^2}{16}, \quad 0 \leqslant x \leqslant L. \tag{2}$$

We prefer to use the dimensionless variables ξ and $\eta = \eta(\xi)$ defined as

$$\xi = x/L, \quad 0 \leqslant \xi \leqslant 1, \quad \text{and} \quad \eta = v/L. \tag{3}$$

Then eqn (2) takes the even simpler form

$$\eta = D\xi^2(2\xi^2 - 5\xi + 3), \quad 0 \leqslant \xi \leqslant 1, \quad \text{where } D = qL^3/(48EI). \tag{4}$$

Assuming that the deflection v on the beam should not exceed a critical value $y_0 = \eta_0 L$ (determined, e.g., for strength reasons about the beam), we must have, because of eqn (4),

$$\forall \xi \quad \text{such that} \quad 0 \leqslant \xi \leqslant 1 \quad \Rightarrow \quad P(\xi) = 2\xi^4 - 5\xi^3 + 3\xi^2 - \varepsilon_0 \leqslant 0, \tag{5}$$

where, because of the last of eqns (3) and (4),

$$\varepsilon_0 = \eta_0/D = y_0/(DL) = 48EIy_0/(qL^4). \tag{6}$$

The problem described by the quantified formula (QF) (5) is, essentially, a decision problem with parameters. We have used dimensionless variables in order to become able to finally get just one parameter, ε_0, in our fundamental polynomial $P(\xi)$ in the QF (5) instead of five ordinary parameters: E, I, q, L and y_0. This constitutes a great simplification of our problem. Obviously, ε_0 is our fundamental variable in the two-dimensional (ε_0, ξ)-space in the QF (5) (ε_0 being the free variable and ξ the quantified variable in this equation) and, therefore, we will construct below a CAD of the ε_0-axis only. Returning to the QF (5), we easily observe that we do not have only a decision problem, but also a computational quantifier elimination (CQE) problem. This is the case since the QF (5) should hold true *for all* values of ξ, or equivalently, since the same QF contains the symbol $\forall$, which is the universal quantifier. In fact, the QF (5) is not very convenient in practice and, therefore, our decision problem is assumed to have been solved provided that we will have been able to derive a *completely* equivalent equation free from $\forall$ (and from ξ), that is a condition containing only the parameter ε_0 in the QF (5). This will be our task below.

In order to evaluate the number of zeros of $P(\xi)$ in $(0, 1)$, we will use the classical tool of Sturm's sequences and Sturm's theorem (Mishra, 1993). These polynomials were constructed by using the commands for the derivative of a function and the remainder in the division of two polynomials, respectively, based on the definitions of a Sturm sequence. Therefore, we easily found the following Sturm sequence $[h_0(\xi), h_1(\xi), h_2(\xi), h_3(\xi), \tilde{h}_4(\xi)]$:

$$\begin{aligned}
h_0(\xi) &= P(\xi) = 2\xi^4 - 5\xi^3 + 3\xi^2 - \varepsilon_0,\\
h_1(\xi) &= h_0'(\xi) = 8\xi^3 - 15\xi^2 + 6\xi,\\
h_2(\xi) &= -\text{remainder}[h_0(\xi), h_1(\xi), \xi] = \frac{27}{32}\xi^2 - \frac{15}{16}\xi + \varepsilon_0,\\
h_3(\xi) &= -\text{remainder}[h_1(\xi), h_2(\xi), \xi] = \left(\frac{256}{27}\varepsilon_0 + \frac{64}{81}\right)\xi - \frac{1760}{243}\varepsilon_0,\\
\tilde{h}_4(\xi) &= -\text{remainder}[h_2(\xi), h_3(\xi), \xi] = -\frac{9\varepsilon_0(2048\varepsilon_0^2 - 117\varepsilon_0 - 108)}{128(12\varepsilon_0 + 1)^2}.
\end{aligned} \tag{7}$$

Since $\tilde{h}_4(\xi)$ is a fraction, we have used its numerator $h_4(\xi)$ and its denominator $h_5(\xi)$ instead.

The critical points for the decomposition of the ε_0-axis are the zeros of the above h-polynomials for $\xi = 0$ as well as for $\xi = 1$. These zeros were easily found to be

$$\varepsilon_{0,1} = -6/17 \approx -0.352941, \quad \varepsilon_{0,2} = (117-165\sqrt{33})/4096 \approx -0.202845,$$
$$\varepsilon_{0,3} = -1/12 \approx -0.083333, \quad \varepsilon_{0,4} = 0, \quad \varepsilon_{0,5} = 3/32 \approx 0.093750,$$
$$\varepsilon_{0,6} = (117+165\sqrt{33})/4096 \approx 0.259974. \tag{8}$$

In this way, we have constructed the following CAD of the ε_0-axis with thirteen elements:

$$D_0 = [(-\infty, \varepsilon_{0,1}), \{\varepsilon_{0,1}\}, (\varepsilon_{0,1}, \varepsilon_{0,2}), \{\varepsilon_{0,2}\}, (\varepsilon_{0,2}, \varepsilon_{0,3}), \{\varepsilon_{0,3}\},$$
$$(\varepsilon_{0,3}, \varepsilon_{0,4}), \{\varepsilon_{0,4}\}, (\varepsilon_{0,4}, \varepsilon_{0,5}), \{\varepsilon_{0,5}\}, (\varepsilon_{0,5}, \varepsilon_{0,6}), \{\varepsilon_{0,6}\}, (\varepsilon_{0,6}, \infty)]. \tag{9}$$

As far as the six "point" elements of this CAD, D_0, are concerned, we can directly use the commands for Sturm sequences and determine the number of distinct real zeros of $P(\xi)$ in the QF (5) and the first of eqns (7) inside the interval $(0, 1)$. Furthermore, for the seven "interval" elements of D_0, we can choose just one sample point in each one of these intervals and work only with these concrete points. Our conclusions will then be valid for the whole respective interval. This is a main advantage of the CAD method for CQE, which essentially permits us to replace the parameter ε_0 in the QF (5) by just few concrete points.

The results that we obtained revealed that $P(\xi)$ has no distinct real zeros in the "composite" intervals (clusters in the CAD notation, since they consist of more than one cell) $(-\infty, \varepsilon_{0,4} = 0)$ and $(\varepsilon_{0,6}, \infty)$, it has just one such zero at the points $\varepsilon_{0,4} = 0$ and $\varepsilon_{0,6}$ and it has two such zeros in the interval $(\varepsilon_{0,4} = 0, \varepsilon_{0,6})$. Therefore, the QF (5) holds true for

$$\varepsilon_0 \in [\varepsilon_{0,6}, \infty), \quad \text{with}\, \varepsilon_{0,6} \approx 0.259974, \tag{10}$$

only, as can easily be observed. This result, eqn (10), constitutes the QFF completely equivalent to the QF (5) in the present elementary, one-dimensional, structural mechanics CAD application. Thus we have been able to find a condition, including only the parameter ε_0, so that we can be sure that the deflection of the beam does not exceed y_0. By taking into account eqns (6), we observe that the QFF (10) contains all five parameters in the present beam problem. Now we will proceed to a more complicated CAD application.

4. A TWO-DIMENSIONAL ELASTICITY PROBLEM

Here we will consider a generalization of a simple stress problem already having been studied (with the elementary approach of Sturm's sequences) by Ioakimidis (1996a). We assume that we have a circular isotropic elastic region G (of radius R) in the OXY-plane

$$G := X^2 + Y^2 - R^2 \leqslant 0, \tag{11}$$

appropriately loaded in such a way that the biharmonic stress component σ_y in the whole region G is given by

$$\sigma_y = A(X^2 + XY + Y^2) \tag{12}$$

(where A denotes a parameter) for all points (X, Y) of the elastic medium G. We can add that this formula for σ_y arises if the biharmonic Airy stress function $\phi(X, Y)$ has an appropriate form of a quartic polynomial in X and Y (Timoshenko and Goodier, 1970).

Our problem is to find whether σ_y does not exceed a critical, failure-related value Σ_0 in the whole circular region G or not. Therefore, we have to check whether

$$(\forall X)(\forall Y) \quad \text{such that} \quad X^2+Y^2 \leqslant R^2 \quad \Rightarrow \quad \sigma_y = A(X^2+XY+Y^2) \leqslant \Sigma_0 \tag{13}$$

or not. If the quantified formula (QF) (13) holds true, we assume that we are safe as far as the problem of the failure of the elastic medium, due to σ_y, is concerned. We wish to decide about the validity of the QF (13). We observe from this formula that we have three parameters (free variables): the radius R of the elastic medium, the proportionality constant A, a kind of "intensity" of σ_y, and the maximum permissible value Σ_0 of σ_y in our whole region G.

Here, because of the appearance of the aforementioned three parameters (a geometric one, R, a loading one, A, and a strength/failure one, Σ_0), we are not able to decide in an absolute manner about the truth or the falsity of the QF (13) any more, but, rather, we will "transform" this QF into a completely equivalent condition, the truth (or falsity) of which will have as a direct consequence the truth (or falsity) of the QF (13) and vice versa. This condition will contain only the three parameters in the QF (13), not the main, quantified variables X and Y and the related universal quantifier $\forall$. Therefore, our condition will be a quantifier-free formula (QFF). The lack of the quantifier $\forall$ in this condition will permit us to use it quite easily contrary to the QF (13), which needs some algebraic skill in order to be used for the solution of our decision problem. We can also add that for the present CQE problem we will use again the CAD method, but here we will have two more variables in the constructed CAD and we will not essentially use Sturm's sequences.

At first, it is obviously wise to simplify the QF (13) a little. By introducing the dimensionless variables

$$x = X/R, \quad y = Y/R \quad \text{and} \quad s = \Sigma_0/(AR^2), \tag{14}$$

we have been able to reduce the number of independent parameters in our CQE problem (13) from three to just one. Then this problem can be rewritten as

$$(\forall x)(\forall y) \quad \text{such that} \quad P_1(s,x,y) = x^2+y^2-1 \leqslant 0 \quad \Rightarrow \quad P_2(s,x,y) = x^2+xy+y^2-s \leqslant 0, \tag{15}$$

where we have two polynomials

$$P_1(s,x,y) = x^2+y^2-1 \quad \text{and} \quad P_2(s,x,y) = x^2+xy+y^2-s \tag{16}$$

and three variables, s, x, and y, the first of which, s, is our overall parameter.

In the formed "three-dimensional" (s, x, y)-space, we will apply CAD and we will decompose this space into appropriate cells. At first, we can "eliminate" y from $P_{1,2}(s,x,y)$ or, better, project the aforementioned three-dimensional space onto the two-dimensional space (s,x). This can be done easily in the present application by using only discriminants and resultants. The discriminants are used for finding the projections of the cylinders $P_1(s,x,y) = 0$ and $P_2(s,x,y) = 0$ onto the (s,x)-plane, whereas the related resultant permits the elimination of the variable y between these two polynomials. Thus we found the three polynomials:

$$\begin{aligned}
Q_1(s,x) &= \text{discriminant}[P_1(s,x,y),y] = -4x^2+4 = -4(x+1)(x-1),\\
Q_2(s,x) &= \text{discriminant}[P_2(s,x,y),y] = -3x^2+4s,\\
Q_3(s,x) &= \text{resultant}[P_1(s,x,y),P_2(s,x,y),y] = x^4-x^2+(s-1)^2.
\end{aligned} \tag{17}$$

These polynomials are the polynomials which will be used below in order to construct the two-dimensional CAD of the (s,x)-plane only. Moreover, since $Q_{1,2,3}(s,x)$ refer only to a plane, their graphical representation is very easy. In fact, for convenience, we have drawn

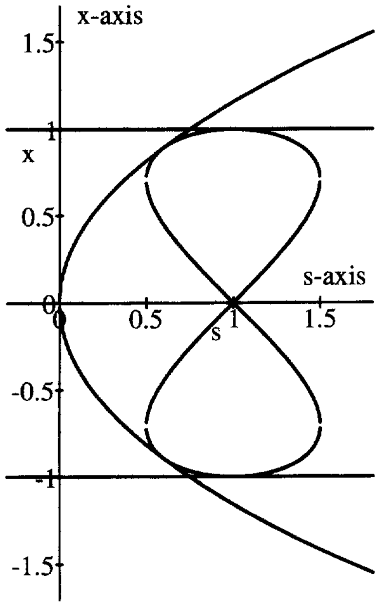

Fig. 1. The projection of the three-dimensional (s, x, y)-space onto the two-dimensional (s, x)-space (based on $Q_{1,2,3}(s, x)$).

$Q_{1,2,3}(s, x)$ in Fig. 1. By taking into consideration this figure, we can easily study the two-dimensional projection of the present (s, x, y)-space onto the (s, y)-space.

Now we will proceed to a second projection, this time from the two-dimensional (s, x)-space onto the two-dimensional s-space, equivalently, the real s-axis. This will be done again by using the computer although, in our elementary application, this second projection can also easily be made by "hand and pencil". At first, we notice that $Q_1(s, x)$ does not depend on s and, therefore, we cannot project it onto the s-axis. Next, $Q_2(s, x)$ is a simple parabola and it leads easily to the new polynomial

$$R_1(s) = \text{discriminant}[Q_2(s, x), s] = 48s. \tag{18}$$

We consider now the quartic polynomial $Q_3(s, x)$ in the third of eqns (17). It is well known that the number of distinct real zeros of the quartic polynomial

$$S(x) = x^4 + qx^2 + rx + t \tag{19}$$

is governed by the signs of q as well as of the quantities (Arnon and Mignotte, 1988)

$$L = 8qt - 9r^2 - 2q^3, \quad \Delta = 256t^3 - 128q^2t^2 + 144qr^2t + 16q^4t - 27r^4 + 4q^3r^2, \tag{20}$$

the second of which is the discriminant of $S(x)$. As is clear from the third of eqns (17) and eqn (19), $q = -1$, $r = 0$ and $t = (s-1)^2$. Therefore, eqns (20) yield

$$R_2(s) \equiv L = -2(2s-1)(2s-3), \quad R_3(s) \equiv \Delta = 16[(s-1)(2s-1)(2s-3)]^2. \tag{21}$$

Finally, three more R-polynomials were constructed from the resultants of $Q_{1,2,3}(s, x)$:

$$R_4(s) = \text{resultant}[Q_1(x, s), Q_2(x, s), x] = 16(4s-3)^2,$$

$$R_5(s) = \text{resultant}[Q_1(x, s), Q_3(x, s), x] = 256(s-1)^4,$$

$$R_6(s) = \text{resultant}[Q_2(x, s), Q_3(x, s), x] = (5s-3)^4. \tag{22}$$

The zeros of $R_{4,5,6}(s)$ concern the points of intersection of $Q_{1,2,3}(s, x)$ (Fig. 1).

Algebraically, we can now easily find the distinct real zeros of the polynomials $R_j(s)$ $(j = 1, 2, \ldots, 6)$. The ordered list of these zeros (multiple zeros are ignored) is

$$S = \left[0, \frac{1}{2}, \frac{3}{5}, \frac{3}{4}, 1, \frac{3}{2}\right] = [0.000, 0.500, 0.600, 0.750, 1.000, 1.500]. \tag{23}$$

Therefore, we have six isolated points in the CAD of the s-axis and seven related open intervals, five of which are finite and two semi-infinite. In these intervals, we must now select just one sample point, e.g., the points of the ordered list

$$S^* = [-0.250, 0.250, 0.550, 0.675, 0.875, 1.250, 1.750] \tag{24}$$

and, in this way, we have the following thirteen concrete points on the s-axis, which describe completely what is taking place in our problem

$$S_0 = [-0.250, 0.000, 0.250, 0.500, 0.550, 0.600, 0.675,\\ 0.750, 0.875, 1.000, 1.250, 1.500, 1.750]. \tag{25}$$

We should also take into account that for each one of the seven sample points in S^* for the intervals in the CAD of the s-axis, we must know the corresponding interval that this sample point represents. These intervals are obvious from eqn (23).

For all these concrete s-points, we should now follow the inverse way in the CAD of the second projection of our P-polynomials. This second projection is clear from Fig. 1. The related computations were made automatically inside the computer. Here we describe the results only for $\tilde{s} = 5/4 = 1.250$ in eqns (24) and (25) corresponding to the s-interval $L_0 = (1.000, 1.500)$. In this case, at first we have to find the corresponding CAD of the vertical x-line in the (s, x)-plane by determining the distinct real x-zeros of the polynomials $Q_{1,2,3}(x, s)$ for $s = \tilde{s}$. These zeros were seen to be eight and they form the ordered list

$$X \approx [-1.29099, -1.00000, -0.96593, -0.25882, 0.25882, 0.96593, 1.00000, 1.29099]. \tag{26}$$

This list should be supplemented by additional sample points representative of the intervals formed by the isolated points in X. These additional points can be the midpoints in the seven finite intervals determined from the isolated points in X together with two more sample points for the semi-infinite intervals defined by X. Then we finally get the seventeen points.

$$X_0 \approx [-2.29099, -1.29099, -1.14550, -1.00000, -0.98296, -0.96593,\\ -0.61237, -0.25882, 0.00000, 0.25882, 0.61237, 0.96593,\\ 0.98296, 1.00000, 1.14550, 1.29099, 2.29099]. \tag{27}$$

For the other s-points in S_0, we must work in a similar manner.

Next, we have to work with the original P-polynomials in eqns (16), considered now as functions of y only in the three-dimensional (s, x, y)-space. For every pair of (s, x)-points in the two-dimensional (s, x)-space (where s belongs to S_0 and x to the appropriate $X_0(s)$, corresponding to the particular s under consideration, e.g., to X_0 in eqn (27) for $s = \tilde{s} = 1.250$), we must now solve $P_{1,2}(s, x, y)$ with respect to y and determine the list Y of the zeros of both these polynomials. As a simple example, in the case where

$$(s, x) = (\tilde{s}, \tilde{x}) \approx (1.25000, 0.61237), \quad \tilde{s} \in S_0, \quad \tilde{x} \in X_0, \tag{28}$$

we easily found the following ordered list:

$$Y \approx [-1.29044, -0.79057, 0.67806, 0.79057], \tag{29}$$

defining three finite and two semi-infinite intervals in the y-direction. By adding appropriate sample points in all these five intervals, we find the whole ordered list of the y-points

$$\begin{aligned} Y_0 \approx [&-2.29044, -1.29044, -1.04050, -0.79057, \\ &-0.05625, -0.67806, 0.73432, 0.79057, 1.79057]. \end{aligned} \tag{30}$$

Therefore, to the $(\bar{s}, \bar{x})$-pair in eqn (28) there correspond nine cells. Let us consider these cells in more detail as far as the signs of the original polynomials $P_{1,2}(s, x, y)$ in the QF (15) and eqns (16) are concerned (with $\mathrm{sign}(0) \equiv 0$). These signs are obviously invariable for all of the points in each cell. Calling these particular points M_i ($i = 1, 2, \ldots, 9$), we easily reach the following conclusions, where the ordered pairs p_i denote the signs of $P_{1,2}(s, x, y)$:

$$\begin{aligned}
M_1 &\approx (+1.250, +0.612, -2.290) &&\Rightarrow\quad p_1 = [+1, +1], \\
M_2 &\approx (+1.250, +0.612, -1.290) &&\Rightarrow\quad p_2 = [+1, 0], \\
M_3 &\approx (+1.250, +0.612, -1.040) &&\Rightarrow\quad p_3 = [+1, -1], \\
M_4 &\approx (+1.250, +0.612, -0.791) &&\Rightarrow\quad p_4 = [0, -1], \\
M_5 &\approx (+1.250, +0.612, -0.056) &&\Rightarrow\quad p_5 = [-1, -1], \\
M_6 &\approx (+1.250, +0.612, +0.678) &&\Rightarrow\quad p_6 = [-1, 0], \\
M_7 &\approx (+1.250, +0.612, +0.734) &&\Rightarrow\quad p_7 = [-1, +1], \\
M_8 &\approx (+1.250, +0.612, +0.791) &&\Rightarrow\quad p_8 = [0, +1], \\
M_9 &\approx (+1.250, +0.612, +1.791) &&\Rightarrow\quad p_9 = [+1, +1].
\end{aligned} \tag{31}$$

From the above sign pairs, at first we observe a continuous change between each p_i and the subsequent one, p_{i+1}. Next, we directly conclude that the only points M_5, M_6 and M_7 lie inside our circular region G, whereas the points M_4 and M_8 lie on the circumference of this circular region. The remaining points, M_1, M_2, M_3 and M_9, lie outside the circular region G and, therefore, they can be neglected. Next, from the second signs in the above sign pairs, we conclude that for the points M_4 and M_5 of our elastic region G, the stress component σ_y is less than its critical value Σ_0, whereas for the point M_6 it reaches the critical value and for the points M_7 and M_8 it exceeds this value.

We will not proceed to further such details on a cell-by-cell basis, which are quite similar to the above ones. In fact, CAD has been able to construct many hundreds of cells in the present application and we can easily decide whether a cell inside the elastic region G and on its boundary satisfies the restriction (15) or not. Although we did not use exact algebraic numbers, but rather their floating-point approximations, we had no essential difficulty to find the points where our P-polynomials in the QF (15) and eqns (16) vanish. This was achieved simply by assuming that a polynomial vanishes if its absolute value takes a very small value depending on the accuracy of the computer software.

Concluding this application, we display in Figs 2, 3 and 4 the relative positions of the circular region G and the elliptical region E for $s = 1.250$, 1.500 and 1.750, respectively, where these three s-points belong to S_0 in eqn (25). The second of these s-points is also a "critical" point of the one-dimensional CAD of the "parametric" s-axis, whereas the first and the last of the above s-points correspond to intervals of the s-axis, more explicitly to $(1.000, 1.500)$ and $(1.500, \infty)$, respectively. These figures were directly drawn on the basis of eqns (16). Additional figures on the "real" (x, y)-plane, corresponding to further values of the parameter s from S_0, either "point" (isolated) values ($s \in S$) or "interval" values ($s \in S^*$), were also drawn, but not displayed here for the sake of space. We restrict ourselves to mention the important "critical" value $s = 1/2 = 0.500$, belonging to both S and S_0, for which G and E are again in contact at two points of their circumferences, but this time with

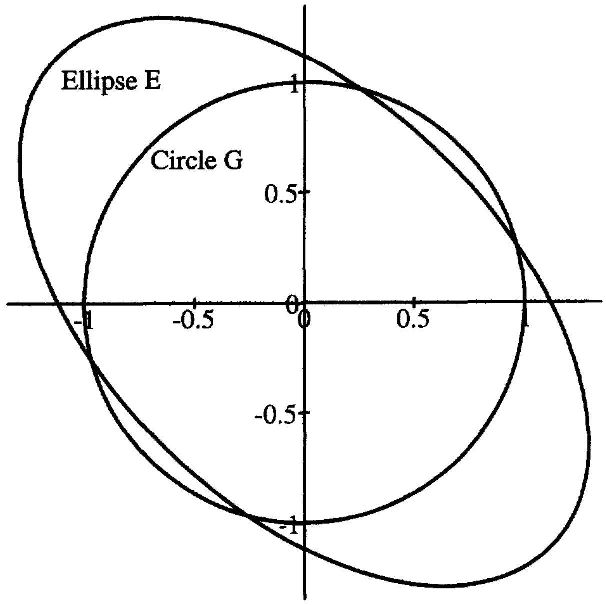

Fig. 2. Relative positions of the circular region G and the elliptical region E for $s = 5/4 = 1.250$ (unfavourable case).

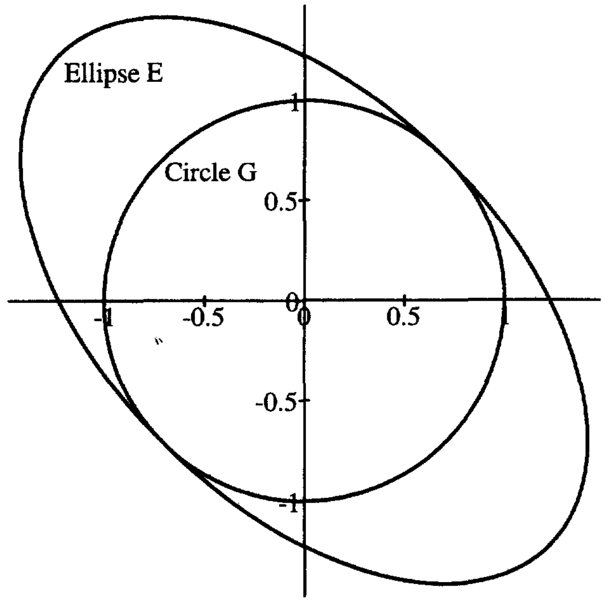

Fig. 3. Relative positions of the circular region G and the elliptical region E for $s = 3/2 = 1.500$ (critical case).

E lying completely inside G. From Figs 2, 3 and 4 we conclude directly that for $s = 3/2 = 1.500$ the boundaries of the circular region G and the elliptical region E are in contact at just two points, for $s = 5/4 = 1.250$ these boundaries intersect at four points and for $s = 7/4 = 1.750$ the boundary of the ellipse E lies outside the boundary of the circle G. These conclusions are completely reasonable from the physical point of view and, what is also important, they result directly from the constructed CAD of the three-dimensional (s, x, y)-space. In passing, we can also add that the aforementioned conclusions about the cells in eqns (31) are quite clear if we take into account Fig. 2 (for $s = \bar{s} = 1.250$) although the same conclusions were drawn above by using the CAD method itself and not just a related figure.

The final conclusion of the present CAD is that the QF (15) (or, equivalently, the QF (13)) holds true provided that

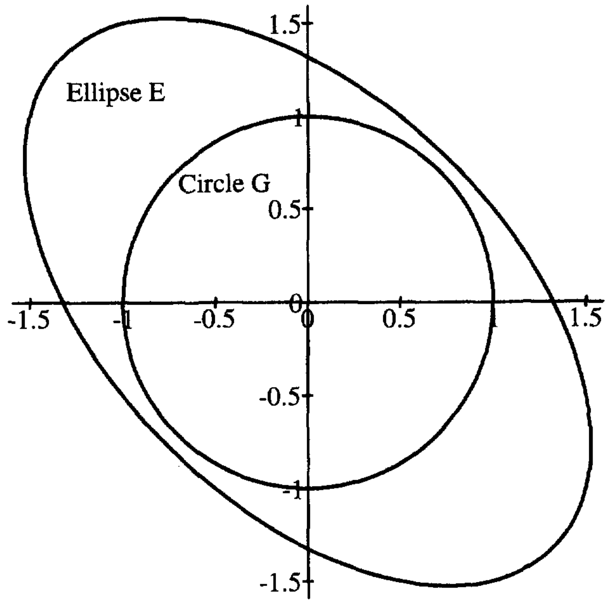

Fig. 4. Relative positions of the circular region G and the elliptical region E for $s = 7/4 = 1.750$ (favourable case).

$$s \geqslant 3/2 = 1.5 \quad \text{equivalently}, \Sigma_0 \geqslant 3AR^2/2 = 1.5AR^2. \tag{32}$$

This is the QFF in the present elementary elasticity CQE problem. Therefore, in order to check the truth or the falsity of the QF (13) during a concrete decision problem, we have just to check the truth or the falsity of the QFF (32), obviously a much simpler task.

5. COMPUTATIONAL DETAILS

5.1. *Computer algebra systems commands*

All of the CAD/QFF results in the previous two sections (both the numerical ones and the symbolic ones) were obtained by using the Maple V computer algebra system and the related commands on a Pentium PC microcomputer under MS-DOS. More explicitly, we performed the necessary computations by using simple procedures based on independent commands. Of course, alternatively, it is completely possible to use any general-purpose computer algebra system although, in rare cases, some useful commands may be missing. We will report below the commands of interest during the derivation of the above QFFs by using (i) Maple V, (ii) Mathematica, and (iii) Axiom.

(i) In Maple V, we have used the commands `diff` for differentiation, `rem` for polynomial remainders, `solve` for the determination of the roots of polynomials (`fsolve` for the numerical determination of these roots), `discrim` for the discriminant of a polynomial, `resultant` for the resultant of two polynomials and `lcoeff` for the leading coefficient of a polynomial. Additional commands (but of less importance) were also used.

(ii) In Mathematica, it is possible to use, instead, the commands `D` for differentiation, `PolynomialRemainder` for polynomial remainders, `Solve` for the determination of the roots of polynomials (appropriately supplemented by `N` for the numerical determination of these roots, alternatively by the purely numerical command `FindRoot`), `Resultant` for the resultant of two polynomials and `Coefficient` (with an appropriate argument) for the leading coefficient of a polynomial. Additional commands can also be used again. Unfortunately, we found no standard command for the discriminant of a polynomial.

(iii) Finally, in Axiom, we can use, instead, the commands `D` for differentiation, `rem` for polynomial remainders, `solve` for the determination of polynomial roots, `discriminant` for the discriminant of a polynomial, `resultant` for the resultant of two polynomials and `leadingCoefficient` for the leading coefficient of a polynomial. Additional related commands can also be used.

We can also add that similar commands are also available in competitive computer algebra systems such as Macsyma and Reduce. Perhaps, it is more interesting to refer to SACLIB (Buchberger *et al.*, 1993), the computer algebra system originating from RISC-Linz, including the `qepcad` package, the most powerful presently available computer implementation of the Collins CAD algorithm (incorporating the improvements by Hong (1990b), partial CAD (Collins and Hong, 1991), etc.). Yet, we feel that we offer no essential service to the reader by reporting the specialized SACLIB commands of interest to the present paper because of their extremely strange names and several variations, completely understood since SACLIB seems to be much more useful to the researcher in computer algebra than to the casual user of its computer-implemented algorithms (such as the present author).

On the other hand, we only found commands for the derivation of Sturm sequences (the `sturmseq` command) and the determination of the number of real roots of a polynomial inside a finite or infinite interval (the `sturm` command) in Maple V. Unfortunately, these two commands do not accept parameters, a fact (to accept parameters) that would be completely reasonable for the first of them. Yet, the interested reader can find a command for the derivation of Sturm–Habicht sequences of polynomials (strongly related to the original Sturm sequences and presenting some nice properties (González *et al.*, 1989)), the `sth` command, in the `IF` package of the share library of Maple V. This non-standard Maple V command accepts any number of parameters contrary to the standard command `sturmseq`.

We believe that the above computational details will be sufficient for the reader who would like to verify and/or extend the results of this paper to further elasticity, structural mechanics, etc. CQE problems. Of course, for this task the best of all is to have accessibility to the `qepcad` package and a related manual. The unrestricted distribution of `qepcad` and the appearance of its manual are expected in the near future. The distribution of the "host" computer algebra system, SACLIB, is already free from RISC-Linz through INTERNET.

5.2. *The influence of floating-point computations*

At this point, we should mention that CAD (and competitive CQE algorithms) traditionally work with exact numbers, that is, generally, with real rational and algebraic numbers during the related computations. This situation assures the correct answer (`true` or `false`) to the original quantified formula (QF) by the related QFF in all cases. On the other hand, in the examples of Sections 3 and 4, we have used floating-point (decimal) approximations to exact numbers. This is convenient from the applied mechanics point of view, since it permits faster numerical computations (in comparison with exact computations), but, unfortunately, in extremely rare cases, it can lead to an incorrect answer (`true` instead of `false` or vice versa). Such an unfortunate situation is very rare and it can be significantly reduced by our employing better floating-point (real) number approximations, e.g. by using high-precision arithmetic inside the computer for real-number computations. (This can be made at will in computer algebra systems). Similarly, we can also improve the accuracy of the additional approximations involved such as the quadrature errors in numerical integration (and further in the boundary and finite element methods), etc. An alternative possibility is to bound the exact numbers that we have to work with by simple rational numbers and use the corresponding bounds, leaded, in this way, to the derivation of two types of QFFs: necessary QFFs and sufficient QFFs, which "bound" the exact QFF and, in practice, should be simultaneously used. In any case, from the practical point of view, it is understood that the present "approximate" CAD approach cannot be guaranteed to always produce completely correct results, but incorrect results appear in extremely rare cases. Furthermore, the whole "inexact" situation can be improved by our reducing the influence of rounding, quadrature and related approximation errors involved in the CAD computations. In spite of these rare cases of a wrong conclusion, it is generally clear that floating-point computations are much faster than exact computations (which may lead to extremely long numerators and denominators in the rational numbers involved or to very complicated algebraic numbers, making no sense to the eye of the engineer) and, moreover, they give us a good indication of what is happening.

On the other hand, in an applied mechanics environment, similar situations (due to approximations) arise quite frequently. For example, the computed value of the critical buckling load P_{cr} for a straight bar under an axial loading is not exact; it is always an approximation to its exact value. Therefore, in a practical problem, where we have an applied axial load P very close to P_{cr}, we may (in rare cases) draw a false conclusion about the buckling (or the avoidance of buckling) in our bar problem. Similarly, in crack problems, both the computed stress intensity factor K at a crack tip as well as its critical fracture-causing bound K_{frac} contain several approximations (due to rounding errors, to quadrature errors, to experimental errors, to errors in the numerical values of the geometry/loading/elastic/fracture parameters involved in the computations, etc.) and, therefore, in extremely rare cases, we may incorrectly decide that we have fracture ($K \geqslant K_{frac}$) of the cracked specimen (or inversely). Modelling errors (e.g. the adoption of classical isotropic elasticity in the above buckling and crack problems, including the acceptance of infinite stress components at the crack tips in the latter case) may also lead to significant further approximations, etc. Under these circumstances, the contribution of the approximations being due to the CAD algorithm itself, where we essentially worked with floating-point (real) numbers (and this permitted us to avail ourselves of significantly faster computations) and not with exact rational/algebraic numbers, essentially constitutes just an extremely small contribution to the reasons which may lead us to an incorrect conclusion in an applied mechanics problem and the same contribution can be reduced at will by using the arbitrary-precision capabilities (in floating-point number computations) of all of the available computer algebra systems. In practice, adopting slightly increased safety factors in the critical quantities involved and appropriately bounding the various related errors will permit us to avoid unpleasant situations in our CAD and additional approximate computations and to reach more conservative results in our applied mechanics problems.

6. MULTI-PARAMETRIC APPLIED MECHANICS PROBLEMS

Although several independent parameters (free variables as opposed to quantified variables) entered into the beam/elasticity applications of Sections 3 and 4 (five parameters: E, I, q, L and y_0 in Section 3 and three parameters: R, A and Σ_0 in Section 4), these parameters were combined just to one (ε_0 in Section 3 and s in Section 4). The consequence of these reductions of the number of parameters has been the rather direct and easy solution of the applied mechanics problems in these sections, essentially by hand together with the precious help of a computer algebra system, but completely interactively. Moreover, the derived quantifier-free formulae (QFFs) ((10) and (32), respectively) were seen to be extremely simple and this had to be more or less expected for essentially single-parametric problems.

On the other hand, although, in most cases, any computational quantifier elimination (CQE) problem actually solved by the CAD algorithm can also be solved manually without using CAD (but, generally, with the help of a computer algebra system), in the applications of Sections 3 and 4, such an approach of solution (not based on CAD) is particularly easy. We will report below such extremely simple methods of solution for the applications of the previous two sections.

(i) At first, with respect to the beam problem of Section 3, we are interested in the non-positivity (on the interval [0, 1]) of the polynomial $P(\xi)$ defined by the QF (5). Now for $\xi = 0$ we must obviously have $\varepsilon_0 \geqslant 0$. Next, for $\xi = 1$ we find the same condition again. (This is evident since the beam was assumed supported at both of its ends.) Therefore, our sole task is to consider the points of extrema of $P(\xi)$ along the beam (on [0, 1]). At these points, the first derivative of $P(\xi)$,

$$P'(\xi) = h_1(\xi) = 8\xi^3 - 15\xi^2 + 6\xi, \tag{33}$$

should vanish. The roots $\xi_{1,2,3}$ of this elementary cubic polynomial can easily be computed:

$$\xi_1 = 0, \quad \xi_{2,3} = (15 \pm \sqrt{33})/16. \tag{34}$$

The condition $\varepsilon_0 \geqslant 0$ corresponding to ξ_1 was already mentioned. Next, as far as the roots $\xi_{2,3}$ are concerned, only the second of them, ξ_3, actually lies in the interval $(0, 1)$ (on the physical beam). In order that $P(\xi)$ can be non-positive at this root, we must have (as is easily computed)

$$\varepsilon_0 \geqslant 3(39+55\sqrt{33})/4096 \approx 0.259974 > 0. \tag{35}$$

Therefore, this our final QFF (coinciding with the QFF (10) in Section 3).

(ii) Secondly, with respect to the elasticity problem of Section 4, it is convenient to move from Cartesian coordinates (x, y) to polar coordinates (r, θ) since the region G under consideration is circular. Then our final CQE problem (15) can be written in the following equivalent form:

$$(\forall r)(\forall \theta) \quad \text{such that} \quad r \in [0, 1], \quad \theta \in [0, 2\pi) \quad \Rightarrow \quad r^2[1+(1/2)\sin(2\theta)] - s \leqslant 0. \tag{36}$$

The worst case is for $r = 1$ (that is at the circumference of the circular region G). For this value of r, we find directly that $s \geqslant 3/2$ (for $\sin(2\theta) = 1$) exactly as has been the case in the QFF (32) already having been derived by the CAD algorithm.

The conclusion from these elementary CQE computations is that CAD, although being a very general algorithm that completely solves a large class of problems, nevertheless, it may be inefficient on simple examples such as the aforementioned applications of Sections 3 and 4. On the contrary, the CAD algorithm seems to be an essential component of the problem solution in applications extremely difficult to perform by hand. Three such rather difficult CQE problems will be reported below in this section on the understanding that:

(i) Even these difficult problems can be solved by hand (with the probable, frequently indispensable, aid of the computer) by the experienced researcher in CQE after sufficient effort. In passing, it can be mentioned that, on the other hand, in few cases, one can similarly derive QFFs in problems where CAD fails to lead to a QFF.

(ii) In difficult CQE problems, one has very complicated CAD computations (very large numbers of projection polynomials, cells, stacks, required computer time, etc.) and, therefore, the display of the intermediate results is completely impossible (contrary to what has been the case in the applications of Sections 3 and 4). Therefore, here we will restrict ourselves to reporting the final QFFs of the three CQE problems to be considered below, all of which will finally include at least three independent parameters instead of just one in Sections 3 and 4.

(iii) These QFFs have already been derived by devoted CAD implementations (mainly by Arnon, Collins, Hong and McCallum), but here we will apply the related results to three concrete applied mechanics (elasticity, fracture mechanics and contact mechanics) problems, which will illustrate the actual practical usefulness of the same QFFs.

(iv) The QFFs to be reported in this section seem to approach the limits of the present possibilities of the CAD algorithm in the sense that for somewhat more complex CQE problems (more explicitly, if one more parameter or inequality is present), the presently available CAD implementations (mainly Hong's `qepcad` SACLIB package) will not be able to reach a QFF. In order to become somewhat more explicit in this comment, we will report, in the next section, Section 7, three examples where `qepcad` seems to be incapable of producing a QFF. Therefore, the results of the present and the next section will essentially "bound" the limits of the CAD algorithm in its present version, giving to the reader a rough idea of what he can expect from CAD (and what he cannot) in his applied mechanics CQE problems.

We will proceed now to the present non-trivial CAD-based applications.

6.1. *A more complex two-dimensional elasticity problem*

The present CQE problem is related to the elasticity problem of Section 4, but it is somewhat more complicated. We consider the elliptical region

$$E := \left(\frac{X-C}{A}\right)^2 + \left(\frac{Y}{B}\right)^2 - 1 \leqslant 0 \tag{37}$$

(instead of the circular region G in Section 4) and the stress component σ_y given by

$$\sigma_y = A_0(X^2+Y^2) \tag{38}$$

(instead of the related expression (12) in Section 4), where A_0 is a positive constant. The above expression for σ_y obviously satisfies the biharmonic equation, $\nabla^4\sigma_y = 0$, as it should in two-dimensional elasticity problems (being a second derivative of the Airy stress function $\phi(X, Y)$). Again we wish that σ_y cannot exceed a critical, failure-related value Σ_0 (assumed positive). Then we have the following CQE problem:

$$(\forall X)(\forall Y) \quad \text{such that} \quad \left(\frac{X-C}{A}\right)^2 + \left(\frac{Y}{B}\right)^2 \leqslant 1 \quad \Rightarrow \quad \sigma_y = A_0(X^2+Y^2) \leqslant \Sigma_0, \tag{39}$$

which is analogous to (but more difficult than) the CQE problem (13) in Section 4.

Now, by defining the new parameter

$$m = \sqrt{\Sigma_0/A_0} \tag{40}$$

as well as the related new, dimensionless variables (both quantified variables and free variables–parameters)

$$x = X/m, \quad y = Y/m, \quad a = A/m, \quad b = B/m, \quad c = C/m \tag{41}$$

and taking into account that it is sufficient that the bound Σ_0 should not be exceeded only at the circumference Γ of the ellipse E so that it is not exceeded in the whole ellipse, we can easily reduce the CQE problem (39) to the following simpler form:

$$(\forall x)(\forall y) \quad \text{such that} \quad \left(\frac{x-c}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \Rightarrow \quad x^2+y^2 \leqslant 1. \tag{42}$$

In this form, our present CQE problem coincides to the well-known x-axis ellipse CQE problem having been extensively studied by CAD (Arnon and Mignotte, 1988; Hong, 1990b, 1992; Collins, 1997). Not entering into many related details, we will report the solution by Hong (1990b, 1992) under the mild assumption that c is non-negative. Then Hong's CAD approach (based on partial CAD) yields the QFF:

$$Q_1 := C_0 \quad \text{and} \quad (b^2-a \leqslant 0 \quad \text{or} \quad b^2c^2+b^4-a^2b^2-b^2+a^2 \leqslant 0), \tag{43}$$

where C_0 denotes the originally assumed and obvious conditions

$$C_0 := 0 < a \leqslant 1 \quad \text{and} \quad 0 < b \leqslant 1 \quad \text{and} \quad 0 \leqslant c \leqslant 1-a. \tag{44}$$

Hong (1990b) explains in great detail the related computational approach and gives accounts of the required projection polynomials, cells, stacks, computer times, etc. both by his own CAD variant (based on qepcad) and by the original Collins' CAD algorithm. The outcome is that the original CAD algorithm cannot reach a QFF in the present restricted ellipse problem contrary to Hong's improved CAD algorithm. The interested reader can consult the thesis by Hong (1990b), but it can be added that Arnon and Mignotte (1988) reached also a QFF, by using Arnon's improvement of the original CAD algorithm, which is very similar to the QFF (43) and, for the sake of space, will not be reported here.

Finally, what seems to be of some interest in our present elasticity application is the rewriting of the QFF (43) (supplemented by the geometrical conditions (44)) in terms of the original variables. Then we easily get the modified QFF

$$Q_1^* := C_0^* \quad \text{and} \quad [B^2 \leqslant A\sqrt{\Sigma_0/A_0} \quad \text{or} \quad B^2(B^2+C^2-A^2) \leqslant (\Sigma_0/A_0)(B^2-A^2)] \tag{45}$$

with

$$C_0^* := 0 < A \leqslant \sqrt{\Sigma_0/A_0} \quad \text{and} \quad 0 < B \leqslant \sqrt{\Sigma_0/A_0} \quad \text{and} \quad 0 \leqslant C \leqslant \sqrt{\Sigma_0/A_0} - A, \tag{46}$$

which assures that σ_y does not exceed its upper bound Σ_0 in the whole ellipse E, simultaneously being both a necessary and a sufficient related condition.

6.2. *A fracture mechanics problem—Sih's strain-energy-density factor*

As a second difficult application of the CAD algorithm, we consider the two-dimensional problem of a crack tip (with the crack lying along the Ox-axis) simultaneously under mode I, II and III conditions. Then a quantity of great importance in fracture mechanics is the intensity S of the strain-energy-density field V (Sih's strain-energy-density factor), defined (asymptotically, for $r \to 0$) by $S = rV$ (with (r, θ) denoting the polar coordinates with centre O the crack tip). It is easily seen that S is determined from (Sih, 1973)

$$S = S(\theta) = a_{11}k_1^2 + 2a_{12}k_1k_2 + a_{22}k_2^2 + a_{33}k_3^2, \tag{47}$$

where k_1, k_2 and k_3 refer to the related mode I, II and III (respectively) stress intensity factors at the crack tip (for convenience, having been divided by $\sqrt{\pi}$). Moreover, the coefficients a_{11}, a_{12}, a_{22} and a_{33} in eqn (47) are functions of the shear modulus G and the Poisson ratio ν of the isotropic elastic material as well as of the polar angle θ about the crack tip $(-\pi < \theta < \pi)$ and they are given by (Sih, 1973)

$$\begin{aligned} a_{11} &= (3-4\nu-\cos\theta)(1+\cos\theta)/(16G), \\ a_{12} &= 2\sin\theta[\cos\theta-(1-2\nu)]/(16G), \\ a_{22} &= [4(1-\nu)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)]/(16G), \\ a_{33} &= 1/(4G). \end{aligned} \tag{48}$$

The distinction between plane strain and generalized plane stress (for $k_3 = 0$) has been made in these particular formulae by Gdoutos (1984), who simply appropriately used the Muskhelishvili constant κ (instead of the Poisson ratio ν) in the above equations.

Here we will be interested in the simple, rather educational case where we wish that the global maximum value of S, $S_{\max}$, does not exceed a related critical value S_Y so that yielding can be avoided. This simply means that we wish that the strain-energy density V does not exceed a critical value V_Y along the whole circumference $(-\pi < \theta < \pi)$ of the circle $r = r_c$ with r_c an appropriate polar radius, probably the radius of the core region. Then, obviously, $S_Y = r_cV_Y$. Of course, this problem is based on several hypotheses, but it seems to be of some interest in fracture mechanics, since, according to the second Sih's fundamental hypothesis of the strain-energy-density criterion, failure by yielding about a crack tip occurs when $V_{\max}$ reaches its critical value (Sih, 1981b). (Here, just for computational convenience, we assumed, almost equivalently, that $V_{\max}$ should not exceed this value, V_Y.) Similarly, Gdoutos (1984) also mentions that "the maximum value S_{max} is related to yielding". Of course, it is understood that there exist several theories about failure by fracture or by yielding, some of which may be more accurate than the strain-energy-density criterion (this depending, of course, on the properties of the material used), but here we do not intend to enter into such a discussion although few related possibilities will be reported in Section 8 (point iv).

The above elementary fracture mechanics problem means that in order that failure by yielding can be avoided, we must have

$$S_{max} \leqslant S_Y, \tag{49}$$

which, obviously, can be rewritten as a quantified formula (QF)

$$\forall\theta \quad \text{such that} \quad -\pi < \theta < \pi \quad \Rightarrow \quad S(\theta) \leqslant S_Y \tag{50}$$

so that yielding cannot start on the curve $r = r_c$. This is a CQE problem, which we will try to solve with the help of the CAD algorithm just below.

At first, we notice that the peculiarity of the present fracture mechanics problem is that we have to work with the sine and cosine trigonometric functions instead of a polynomial. But by using the elementary and frequently employed variable transformations

$$\cos\theta = (1-y^2)/(1+y^2), \quad \sin\theta = 2y/(1+y^2) \quad \text{with } y = \tan(\theta/2) \in (-\infty, \infty) \tag{51}$$

and after some simple algebraic computations, we easily find that the inequality $S(\theta) \leqslant S_Y$ in the QF (50) can easily be rewritten as a quartic polynomial in the variable y

$$P(y) = [4S_YG - 2k_2^2(1-v) - k_3^2]y^4 + 4k_1k_2(1-v)y^3 + 2[4S_YG - k_1^2(1-v) + k_2^2 v \\ - k_3^2]y^2 - 4k_1k_2vy + 4S_YG - k_1^2(1-2v) - k_2^2 - k_3^2 \geqslant 0, \quad y \in (-\infty, \infty). \tag{52}$$

Therefore, our CQE problem (50) can be reduced to

$$\forall y \quad \text{such that} \quad -\infty < y < \infty \quad \Rightarrow \quad P^*(y) = y^4 + p_0y^3 + q_0y^2 + r_0y + t_0 \geqslant 0, \tag{53}$$

where $P^*(y) \equiv P(y)/c_0$, c_0 being the coefficient of y^4 in $P(y)$, and p_0, q_0, r_0 and t_0 are the coefficients of the quartic polynomial $P^*(y)$, easily determined from the coefficients of the original polynomial $P(y)$ in eqn (52) (after a division by c_0).

Of course, at this point we must pay the due attention to the sign of c_0, which, obviously, should be plus. Therefore, our first condition C_0 (to supplement the CQE problem (53)) is

$$C_0 := c_0 = 4S_YG - 2k_2^2(1-v) - k_3^2 > 0, \tag{54}$$

which requires sufficiently low values of the stress intensity factors k_2 and k_3 and/or a sufficiently large value of S_Y at the crack tip (as it should be expected) so that yielding can be avoided. Of course, the condition (54) is a necessary but by no means a sufficient condition.

Now, assuming the validity of C_0 in advance, we are ready to work with $P^*(y)$ itself. Since this quartic polynomial contains a y^3-term, it seems convenient (this simply due to the already available related CAD results for the non-negativity of the quartic polynomial) to get rid of it. This task is trivial. We simply have to use the new variable

$$x = y + p_0/4 \quad \Leftrightarrow \quad y = x - p_0/4 \tag{55}$$

in the QF (53). Then we find the equivalent QF

$$\forall x \quad \text{such that} \quad -\infty < x < \infty \quad \Rightarrow \quad P^{**}(x) = x^4 + qx^2 + rx + t \geqslant 0, \qquad (56)$$

where the new coefficients, q, r and t, easily result from the coefficients of $P^*(y)$ in the QF (53). The QF (56) constitutes a classical CQE problem in the literature, having been studied by versions of the CAD algorithm by Arnon and Mignotte (1988), Hong (1990b, 1992) and Collins and McCallum (Collins, 1997). Here we will restrict ourselves to report the related QFF derived by Hong

$$[(\Delta \geqslant 0 \quad \text{and} \quad L \leqslant 0) \quad \text{or} \quad (M \geqslant 0 \quad \text{and} \quad L \geqslant 0)] \quad \text{and} \quad t \geqslant 0, \qquad (57)$$

where L and Δ are given by eqns (20) again, whereas M is given by

$$M = 27r^2 + 8q^3. \qquad (58)$$

Three further simple solutions to the same CQE problem were found by Arnon and Mignotte (1988). All of these solutions include the discriminant Δ of the quartic polynomial $P^{**}(x)$. Most probably, the simplest available QFF for this problem was derived by Collins and McCallum (Collins, 1997). This QFF (also including Δ) is just

$$\Delta \geqslant 0 \quad \text{and} \quad (4t - q^2 \geqslant 0 \quad \text{or} \quad q \geqslant 0). \qquad (59)$$

One further QFF for the non-negativity of the reduced quartic polynomial $P^{**}(x)$, having been derived by McCallum, is also reported by Collins (1997).

Further computational and timing details can be found in the papers by Arnon and Mignotte (1988), Hong (1990b, 1992) and Collins (1997). We can also add that the original CAD algorithm derived a QFF with 401 occurrences of atomic formulae (Hong, 1992), whereas the QFF (59) contains only 3 atomic formulae (compared to 5 in QFF (57)).

From the fracture mechanics point of view, by using the aforementioned CAD results, we have been able to solve the present CQE problem concerning the strain-energy-density factor S near a crack tip and its critical, yielding-causing upper bound S_Y. Our final QFF will consist of both conditions (54) and (57) (or 59)) with the reported intermediate formulae taken also into consideration. This QFF permits us to directly decide about failure by yielding (or not) at the crack tip under consideration for concrete values of the material and stress intensity parameters involved ($G, v, S_Y = r_c V_Y$ and $k_{1,2,3}$) under the assumptions on which the present application has been based.

6.3. *A frictionless punch problem for the elastic half-plane*

As a final application of CAD to applied mechanics, we will consider the classical problem of frictionless contact between a rigid punch of a sufficiently general profile and an isotropic elastic half-plane (with boundary the Ox-axis). The solution to this important contact problem is given, e.g., by Gladwell (1980) under the assumption of complete penetration, that is of complete contact between the punch and the half-plane over the contact region, assumed (without loss of generality) along a finite interval $[-l, l]$ of the real axis (the boundary of the half-plane). Then the dimensionless normal displacement $y(x)/l$ of the half-plane along the contact region $[-l, l]$ will have to match the profile of the punch, which is assumed here to have the form of a cubic polynomial, that is

$$v(\xi) \equiv y(x)/l = a\xi^3 + b\xi^2 + c\xi + d, \quad \xi = x/l \in [-1, 1], \qquad (60)$$

where, for convenience, beyond the dimensionless displacement $v = y/l$, the dimensionless variable $\xi = x/l$ in the contact region has been used as well.

We will take into account the well-known solution to this contact problem (Gladwell, 1980), according to which for a punch profile of the form

$$v(\xi) = \sum_{k=0}^{n} b_k T_k(\xi), \quad n > 0, \quad \xi \in [-1, 1] \tag{61}$$

(with $T_k(\xi)$ denoting the classical Chebyshev polynomial of the first kind and degree k), the normal loading $p_0(\xi)$ of the punch on the half-plane along the contact region $[-1, 1]$ (with respect to ξ) will be given by (Gladwell, 1980)

$$p_0(\xi) = \frac{p(\xi)}{\theta\sqrt{1-\xi^2}} = \frac{1}{\theta\sqrt{1-\xi^2}}\left[q_0 + \sum_{k=1}^{n} k b_k T_k(\xi)\right], \quad \xi \in [-1, 1], \tag{62}$$

where

$$\theta = (1-\nu)/G \quad \text{and} \quad q_0 = \theta P_0/(\pi l) \tag{63}$$

with ν denoting the Poisson ratio and G the shear modulus of the elastic material and, further, P_0 the total load on the half-plane due to the punch (independently of the punch shape).

Equation (62) was derived under the implicit assumption that the punch and the half-plane are in complete contact over the contact region $[-1, 1]$ (Gladwell, 1980), but, clearly, this depends on the coefficients of the punch profile $v(\xi)$ in eqn (61) as well as on the total load P_0, the actual contact interval $2l$ and the elastic constants ν and G of the half-plane. Therefore, by no means can eqn (62) be guaranteed as true in advance. Here our sole aim is to find the necessary and sufficient conditions for a complete contact and, therefore, for the validity of the aforementioned classical elastic solution (62) to the present contact problem.

To this end, we can write the related quantified formula (QF)

$$\forall \xi \quad \text{such that} \quad -1 \leqslant \xi \leqslant 1 \quad \Rightarrow \quad p(\xi) > 0, \tag{64}$$

where the positivity of the square root in eqn (62) has been taken into account in advance. For our assumed punch profile in eqn (60), straightforward computations with the Chebyshev polynomials $T_k(\xi)$ in the above formulae directly reveal that

$$p(\xi) = 3a\xi^3 + 2b\xi^2 + [(-3/2)a + c]\xi - b + q_0, \tag{65}$$

where q_0 is given by the second of eqns (63). Now our sole task is to investigate the conditions for the positivity of the cubic polynomial $p(\xi)$ on $[-1, 1]$. For this task we will apply the related result by Collins (private communication), who used the CAD algorithm (together with the qepcad CAD-based SACLIB package) for the solution to this problem, but on the interval $[0, 1]$.

More explicitly, Collins worked with the polynomial

$$A(t) = a_0 t^3 + b_0 t^2 + c_0 t + d_0, \quad d_0 > 0, \quad t \in [0, 1], \tag{66}$$

reduced it to the polynomial

$$A'(t) \equiv A(t)/d_0 = a'_0 t^3 + b'_0 t^2 + c'_0 t + 1, \quad t \in [0, 1] \tag{67}$$

(with the original coefficients having now been divided by d_0) and found the related QFF

$$a'_0 \neq 0 \quad \text{and} \quad a'_0 + b'_0 + c'_0 + 1 > 0 \quad \text{and} \quad [D' < 0$$
$$\text{or} \quad (b'_0 \geqslant 0 \quad \text{and} \quad c'_0 \geqslant 0) \quad \text{or} \quad (a'_0 + 1 \geqslant 0 \quad \text{and} \quad b'_0 + 2a'_0 \leqslant 0)$$

$$\text{or} \quad (a'_0 - 2 \leqslant 0 \quad \text{and} \quad c'_0 \geqslant 0)], \tag{68}$$

where D' denotes the discriminant of the polynomial $A'(t)$ given by

$$D' = -4a'_0c'^3_0 + b'^2_0c'^2_0 + 18a'_0b'_0c'_0 - 4b'^3_0 - 27a'^2_0. \tag{69}$$

Returning to the original coefficients (a_0, b_0, c_0 and d_0), this QFF takes the following final form (Collins, 1994a):

$$a_0 \neq 0 \quad \text{and} \quad d_0 > 0 \quad \text{and} \quad a_0 + b_0 + c_0 + d_0 > 0 \quad \text{and} \quad [D < 0$$
$$\text{or} \quad (b_0 \geqslant 0 \quad \text{and} \quad c_0 \geqslant 0) \quad \text{or} \quad (a_0 + d_0 \geqslant 0 \quad \text{and} \quad b_0 + 2a_0 \leqslant 0)$$
$$\text{or} \quad (a_0 - 2d_0 \leqslant 0 \quad \text{and} \quad c_0 \geqslant 0)] \tag{70}$$

where D denotes the discriminant of the original polynomial $A(t)$. At this point, we can add that the first condition in this QFF assures just the existence of a cubic polynomial, whereas the next two inequalities are obvious, since they concern the positivity of $A(t)$ at the ends of the interval [0, 1] and, in fact, they were taken into account by Collins (1994a) in advance when be called qepcad through the following input formula (corresponding to the output (68)):

$$(\forall t)[a'_0 + b'_0 + c'_0 + 1 > 0 \quad \text{and} \quad [[t \geqslant 0 \quad \text{and} \quad t \leqslant 1] \quad \Rightarrow \quad a'_0t^3 + b'_0t^2 + c'_0t + 1 > 0]]. \tag{71}$$

In passing, we can add that qepcad required 75 s (on a DECstation 5240 with a RISC processor at 40 MHz) in order to produce the above QFF (68).

This solution by Collins being available, we can directly employ it in our contact problem. We have just to change the contact interval from $[-1, 1]$ to $[0, 1]$ through the elementary variable transformation $\xi = 2t - 1$. Then we find the following cubic polynomial for the polynomial part $p^*(t) \equiv p(\xi)$ of the pressure distribution $p_0(\xi)$ over the contact region:

$$p^*(t) = 24at^3 + 4(-9a + 2b)t^2 + (15a + 2c - 8b)t - (3/2)a + b - c + q_0, \quad t \in [0, 1], \tag{72}$$

which is exactly of the form (66) having been studied by Collins (private communication). Therefore, one can directly use the QFF (70) as the necessary and sufficient condition for a complete contact in the contact region. This QFF consists of 10 atomic formulae compared to just one atomic elementary formula in the QFFs of Sections 3 and 4. Of course, the non-trivial geometry coefficients a, b and c of the punch profile (d simply concerns a rigid displacement), the contact length parameter l as well as the load parameter P_0 and the elastic constants ν and G appear in the present QFF. Moreover, clearly, the above results remain applicable even if $p_0(\xi)$ tends to zero at one or both ends of the contact interval $[-1, 1]$.

Incidentally, it can be added that Collins derived also a competitive QFF by hand on the basis of the Descartes rule of signs. Moreover, very recently, the appearance of a new related paper on the positivity of polynomials, by Hong and Jakus (1996), also came to our attention. It is really unfortunate that the present useful CAD result cannot be directly generalized to more complicated punch profiles because of limitations of the CAD algorithm as will be explained/illustrated in the next section.

7. LIMITATIONS OF THE CAD ALGORITHM

In the previous section, we applied CAD-derived QFFs to three probably significant applied mechanics problems. Unfortunately, CAD is not a very simple algorithm and, generally, it requires much effort inside the computer in order to reach a QFF although, it is true, the enhancements to the algorithm with Hong's qepcad package and its improvements by Collins and his collaborators made CAD much more friendly (at least as far as the required computer time and the appearance of the derived QFF are concerned).

The x-axis Kahan ellipse problem (used in Section 6.1) and the quartic polynomial problem (used in Section 6.2) are two classical examples (probably, the most classical ones) of successful applications of CAD to difficult CQE problems. The positivity of the cubic polynomial on a finite interval (used in Section 6.3) is not a classical CQE problem (at least, as far as CAD is concerned), but it should also be characterized as a difficult CAD problem. Therefore, all three CQE problems in the previous section should be considered as difficult problems (for CAD) in spite of the relative simplicity of the final QFFs. In this section, we will report three additional CQE problems, but where CAD seems incapable of deriving a solution (QFF). These problems concern:

(i) The complete Kahan's ellipse problem (already reported in Section 6.1), where the centre (C, D) of the ellipse E generally lies outside the x-axis. This means that now in the application of Section 6.1, we have

$$E := \left(\frac{X-C}{A}\right)^2 + \left(\frac{Y-D}{B}\right)^2 - 1 \leqslant 0 \tag{73}$$

instead of the inequality (37), where D is an additional parameter (free variable). Up to now we have not seen a CAD- (even qepcad-) derived QFF for the corresponding CQE problem (in our case described by the QF (39)) although a semi-manually derived QFF for this problem was obtained by Lazard (1988), but with the help of the Macsyma computer algebra system in the symbolic computations involved. In passing, we can add that somewhat analogous is the case with the complete quartic polynomial in the QF (53) (compared to its "reduced" form in the QF (56)), which also seems not directly solved by CAD up to now. Of course, although it can be argued that the complete quartic problem can easily be solved on the basis of the reduced quartic problem (this was also done by Lazard in his semi-manual method (1988), based on Sturm's sequences), this seems not to be the case for the complete Kahan's ellipse problem, where CAD's results are, therefore, incomplete (restricted to the x-axis ellipse problem).

(ii) A second problem where CAD failed to produce a QFF is the problem of the positivity of the complete quartic polynomial on a finite interval, more explicitly on $[0, 1]$. This problem was studied in detail by Collins (private communication) by using the qepcad CAD implementation, who kindly informed the author of his related negative conclusion together with the hope that the solution of this CQE problem might be possible after "several scheduled qepcad improvements". From our point of view, this means that CAD is presently inapplicable to the case of a quartic shape of the punch in the indentation contact problem of Section 6.3 above. Evidently, similar is also the case for any higher polynomial degree of the shape of the punch and, therefore, the special CQE/contact problem already studied and completely solved in Section 6.3 most probably reflects the best presently available CAD's possibilities.

(iii) Finally, CAD (qepcad) failed to produce a solution (QFF) to the following positivity CQE problem: given the bivariate polynomial on the unit square

$$P(x, y) = a_1 + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy, \quad (x, y) \in [-1, 1] \times [-1, 1], \tag{74}$$

decide whether $P(x, y)$ remains continuously positive on the same square. This is essentially a two-dimensional generalization of the positivity problem of Section 6.3 and it seems to be of particular importance for the detection of no-contact regions in three-dimensional unilateral contact problems solved by the classical boundary element method. Collins (private communication) brought to the author's attention that qepcad failed to solve the aforementioned positivity CQE problem (after several related attempts) because "the number of the projection polynomials becomes too large in terms of both space and time requirements". This unfortunate situation was due to the fact that the present CQE problem "has six free variables and two bound variables, a total of eight". These comments by Collins mean that if we used more elementary boundary elements, e.g. such an element with the more linear bivariate polynomial

$$P(x, y) = a_1 + a_2 x + a_3 y + a_4 xy \tag{75}$$

(with a total of six variables), then, probably, qepcad could be expected to derive a solution (although, obviously, this is not sure). Unfortunately, two-dimensional boundary elements with just four nodes are of low computational accuracy and actual practical usefulness.

Under these circumstances, we can conclude that although CAD is a general and powerful algorithm in CQE, unfortunately, there are cases, not extremely complex, where it fails to produce a QFF. This means that further improvements in the algorithm, both theoretical and inside its present computer implementation, qepcad (a package of the SACLIB computer algebra system), will be required so that CAD can solve essentially more difficult CQE problems than those in the previous section. On the other hand, presently, CAD remains the most powerful general-purpose CQE algorithm and the solution of moderately difficult CQE problems (such as those in the previous section), of probable importance to applied mechanics, seems to be of sufficient interest both from the theoretical and from the practical point of view.

Now we will proceed to briefly describe further applied mechanics problems, where CAD and CQE algorithms in general seem to be of interest (although not always easily applicable).

8. FURTHER POSSIBLE APPLIED MECHANICS APPLICATIONS

We have already applied CAD to five concrete applied mechanics problems in Sections 3, 4 and 6. Moreover, as was already mentioned in Section 1, we have also used three elementary CQE approaches to the derivation of QFFs in few applied mechanics problems. These problems include the contact of a beam and a rigid obstacle parallel to the beam, equivalently the maximum deflection of the beam (Ioakimidis, 1995a, b), the no-contact conditions for the crack edges for a straight (Ioakimidis, 1997a) and a penny-shaped (Ioakimidis, 1996d) crack, the complete contact of a beam and a tensionless Winkler foundation either along the whole beam (Ioakimidis, 1996c) or with the help of beam finite elements (Ioakimidis, 1996e), inequality constraints in elementary rectangular boundary/finite elements (Ioakimidis, 1996b) and an existential problem for an edge crack in fracture mechanics (Ioakimidis, 1997b). In several of these problems, the much more systematic CAD algorithm is also applicable.

In this section, we will report few further possibilities of application of CAD and CQE methods in general to applied mechanics. The present list of additional applications simply consists of related examples and by no means is it complete or systematic or even classified. A very large number of further possible applications can be suggested quite easily.

(i) CAD/CQE can also be used in problems concerning maximum absolute values for the stress components as has been already the case in the plane elasticity problem of Section 4 (for a circular region) and Section 6.1 (for an elliptical region). Any stress component may be subjected to such constraints. But, in most cases, we are interested in a combination of stress components such as the combination

$$\Sigma = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \tag{76}$$

entering into the classical von Mises yield criterion in three-dimensional elasticity. Quite frequently, we wish that the solution of an elasticity problem should be such that no point of the elastic medium exceeds the elastic limit and enters into the state of plastic deformation. Several such examples in beam problems, torsion problems, plane elasticity problems, etc.

are well known in applied mechanics. CAD can be successfully used for the derivation of QFFs including only the parameters involved (geometric, material, loading and strength parameters) in such a way that we can know whether some concrete point of the elastic specimen has entered into the plastic deformation state or not. (Obviously, our elasticity solution is applicable if no plastic deformation is present.)

(ii) Somewhat analogous is the case of stress concentration factors (e.g. about holes or inclusions) and stress intensity factors (e.g. about crack and V-notch tips, in the latter case in a generalized sense) in elasticity/fracture mechanics problems under reasonable constraints among the parameters involved. Quite frequently, we are interested that these factors, in a real elasticity problem, do not exceed concrete maximum values known in advance. CAD can be used as a convenient tool in the design environment for the derivation of appropriate QFFs, which can further be employed as a check about the acceptability/lack of acceptability of the values of the parameters involved in a concrete application. Quite similar is the case in any problem where an important quantity should not exceed a critical value as has been the case, e.g. in the fracture mechanics problem of Section 6.2.

(iii) CAD can also be used in order to derive QFFs for parametric problems concerning the possibility that a closed contour lies inside (or outside) a simpler closed contour. Such a problem may arise, e.g., in fracture mechanics with the caustics formed on the screen about a crack tip (Theocaris, 1981). These caustics have rather complicated shapes and, therefore, it is interesting to find simpler closed curves (e.g. the circumference of a circle or of an ellipse) so that the caustic lies completely inside this new curve. If this is the case, then the stress intensity factors and related factors determined through the measurement of appropriate dimensions of the caustic could be bounded by estimating appropriate dimensions of the simpler curve inside which the caustic lies. Of course, this approach may be of some practical interest only when the caustic depends on one (or more than one) parameter. Somewhat similar is the case with other curves of interest in elasticity problems (with parameters) such as the curves bounding regions where a stress, displacement or strain component reaches a critical value in a plane elasticity problem or, a more important application, the curve which is the boundary between the elastically and the plastically deformed region of a plane medium. CAD can be used as a tool for the study of some geometric properties of these curves. Similar is the case with closed surfaces in three-dimensional elastic media.

(iv) It is also evident that CAD can also be used for the generalization of the CQE results of Section 6.2, concerning yielding (better the avoidance of yielding) along a circular curve $r = r_c$ about a crack tip, to a series of related fracture mechanics problems such as the problems concerning planar three-dimensional cracks, cracks in anisotropic/composite media, along bimaterial interfaces, in plates and shells, etc. Moreover, beyond the maximum value of the strain-energy-density factor S, S_{max}, its minimum value, S_{min} is also of much interest, since it concerns stable fracture initiation (Sih, 1973, 1981a, b; Gdoutos, 1984), but the related CQE problem seems not easily solvable by CAD, since just a local (not a global) minimum of S should be used. Beyond the papers by Sih (1973, 1981a, b) and the book by Gdoutos (1984), the interested reader can also consult a long series of related papers available in the literature, e.g. some papers in the conference proceedings on mixed mode crack propagation edited by Sih and Theocaris (1981), especially in the first part of this volume. Of course, CAD can also be used in several more similar and dissimilar cases concerning yielding/fracture initiation where Sih's strain-energy-density S-criteria have been substituted by other yielding/stable fracture initiation criteria either simpler or more complicated. For example, such a criterion is the interesting T-criterion, suggested by Theocaris and Andrianopoulos (1982), where the core region is assumed not to be circular any more, but that corresponding to the Mises elastic–plastic boundary (Broek, 1974). Another, rather simple possibility concerns the problem of finding a QFF so that the plastic region about a crack tip (resulting e.g. by using the classical von Mises yield criterion, somewhat different from the strain-energy-density S-criterion having been adopted in Section 6.2, or even the Tresca related criterion (Broek, 1974)) does not reach a circular curve $r = r_c$ (is completely included in the open circle $r < r_c$), a problem somewhat analogous to the Kahan's ellipse problem of Section 6.1.

(v) Another interesting applied mechanics problem concerns the sign of the internal stress components in plane elasticity. Such an investigation is important in cases of materials weakly resistant to compression or, inversely, to tension. Then we must have stress components of the same sign (either plus or minus) at all of the points of the elastic medium. This possibility has been investigated in detail by Bennati *et al.* (1993) for a circular disk and a circular ring under appropriate loadings. For example, restricting ourselves to the circular ring problem studied in this reference, we can easily formulate and solve it as a CQE problem by using CAD.

(vi) Sufficiently analogous is the case with the sign of the normal deflection $v(x, y)$ of a loaded/bent plate B, where, if we are interested in the non-negativity of this deflection (with respect to the plane assumed defined by the boundary of B), we have the CQE problem

$$(\forall x)(\forall y) \quad \text{such that} \quad (x, y) \in S \quad \Rightarrow \quad v(x, y) \geqslant 0. \tag{77}$$

From the well-known solution to the problem of a circular plate loaded by a concentrated normal force P on its centre (in the framework of the related elementary Kirchhoff theory), either simply supported or clamped along the circumference of the circle (Timoshenko and Woinowsky-Krieger, 1959), it is clear (after elementary thoughts based on calculus) that the QFF related to the QF (77) is simply `true`. This happens independently of the values of the geometric, loading and material parameters entering into the related deflection equation and is, more or less, expected. Competitive and much more difficult plate problems can also, probably, be studied as CQE problems on the basis of the QF (77) if the non-negativity of the deflection $v(x, y)$ is of interest. On the other hand, for rectangular plates rigidly clamped along their boundaries and loaded also by a normal concentrated force P, it is well-known (Duffin, 1948) that it is possible to simultaneously have normal displacements $v(x, y)$ both positive and negative (of direction opposite to that of the concentrated force P). This is the famous old Hadamard problem for elastic plates (Duffin, 1948) and it would be very welcome to know whether this minor "paradox" actually takes/does not take place by solving the related CQE problem defined by the QF (77). Obviously, in this problem, the solution will not be simply `true`, but, rather, it will include several of the parameters involved (geometric, loading and material), probably, mainly, the ratio of the dimensions of the square plate.

In principle, there is an infinity of problems arising in applied mechanics, where CQE techniques (concerning either the universal or the existential quantifier or both) are of interest. The applications of Sections 3, 4 and 6 and the above examples concern only few such possibilities and involve inequality constraints. Such constraints appear also in structural optimization problems (Haftka and Gürdal, 1992; Kirsch, 1993), where CQE algorithms are, in principle, applicable in the case of appearance of parameters and, mainly, for the existential CQE problem, that is for the derivation of the necessary and sufficient conditions assuring the existence of a feasible region so that all of the inequality constraints (frequently accompanied by equality constraints) are simultaneously valid there.

Concluding this section, we would like to make also reference to another class of applied mechanics problems where inequality constraints are present. This class concerns the variational inequalities, sufficiently popular in applied mechanics and additional engineering fields (Kinderlehrer and Stampacchia, 1980; Chipot, 1984; Hlaváček *et al.*, 1988; Kikuchi and Oden, 1988; Antes and Panagiotopoulos, 1992). We feel that CQE techniques might be of interest to this class of problems as well, especially to unilateral contact problems. For such contact problems, we have the classical linear complementarity problem, where the boundary condition is that either the contact pressure or the contact gap must be equal to zero at every point of the assumed contact region (with, generally, partial contact) and, simultaneously, both of these quantities should be non-negative. Friction can also be present and the possible contact of the crack faces (crack closure phenomenon) also belongs to unilateral contact problems. The interested reader can consult many related papers (see, e.g., the very recent ones by Kim and Kwak (1996), Lee (1996) and Kalliontzis *et al.* (1996)).

We believe that the above examples of application of CAD/CQE algorithms in general to the derivation of QFFs are more or less sufficient so that the possible impact of these algorithms on applied mechanics can be appreciated.

9. CONCLUSIONS—DISCUSSION

In this paper, we used CAD for the derivation of QFFs in applied mechanics problems involving parameters (free variables) beyond the quantified variables. In Sections 3 and 4, we proceeded to two such concrete applications of CAD and in Section 6 we solved three more complex CQE problems although CAD is, most probably, inefficient in the CQE problems of Section 7. Furthermore, in Section 8 we reported some additional applied mechanics CQE-related problems, where CAD/CQE algorithms are probably applicable.

Our first conclusion is that CAD really constitutes an interesting tool in applied mechanics and it can lead to important QFFs of sufficient practical significance. We hope that CAD will be used in the future in more complicated or even in more challenging applied mechanics problems (e.g. with the substitution of the criterion for the critical value of only one stress component in Section 4 and Section 6.1 by the von Mises yield criterion). In any case, the present results indicate that the CAD algorithm in computer-aided algebra should not be ignored and the computer algebra systems need not be used only for the computation of well-known mathematical objects (such as derivatives, integrals, series, sums, products, etc.).

We can also add that as is well known from the sufficiently extensive literature on CAD, this method is not very easy to use and, in any case, it is not a panacea. In fact, it is a rather complicated method and we cannot have a lot of variables in it. The device frequently employed (as has been the case here, in Sections 3 and 4) is to use, as much as possible, overall parameters and/or reduce the number of parameters by eliminating trivial ones. For example, in the quadratic equation $ax^2+bx+c = 0$, we can easily eliminate a by dividing this equation by a and, therefore, we get just two parameters instead of three during the CAD computations. Evidently, this is completely legitimate only if $a \neq 0$. In practice, the use of a total number of three or four variables is the best that we can expect from CAD if a simple QFF has to be easily found.

Several points of inconvenience when using CAD (together with the related suggested remedies) can be found in the literature. We have already made reference to the possibility of combining or eliminating parameters. Another extremely discussed possibility is the combination of simple cells into clusters so that much less sample points can be used. At the danger of being accused to have misunderstood CAD's fundamental philosophy and insulted computer algebra systems/algorithms, we dare suggest also to use only the "non-trivial" cells in a CAD (that is two-dimensional regions in two-dimensional problems, three-dimensional regions in three-dimensional problems, etc.) and neglect all further cells (of lower dimensions). In this way, CAD's complicated computations would have been considerably simplified and, in fact, the real interest in applied mechanics is just in these cells. Similarly, in this restricted CAD environment, as was already discussed in Section 5.2, we can work only with floating-point numbers, which permits the significant reduction of the computational effort. There are several concrete applications in the CAD references below and full discussions about its capabilities and restrictions. We do not feel it necessary to enter into further computational details, but we can mention again that CAD has been already significantly improved compared to its first version, originally suggested by Collins (1975). Further significant improvements/simplifications of CAD can be found in the very recent book by Caviness and Johnson (1997) especially in the paper by Collins (1997).

We can also add that it would be helpful if the existing modern computer algebra systems were equipped with some kind of algorithm permitting us to solve systems of polynomial inequalities. In this way, it would be possible, e.g., to combine the independent equations of the "good" cells (where the constraints are satisfied) in the one-, two- or multi-dimensional CADs into the appropriate QFFs without much human intervention. This seems not to be the case in Maple V, but, fortunately, the latest version of Mathematica (version 3.0) contains a special standard package for the manipulation and solution of algebraic inequalities.

In any case, wishing to conclude in an optimistic way, we can repeat that CAD and its various variants and simplifications can easily and successfully be used for the solution of CQE problems in applied mechanics if the total number of variables (both free and quantified) does not exceed three or four (as is quite frequently the case). Of course, CAD is not the sole route to this direction, but, surely, it is the most general and feasible one. QFFs permit us to directly get our reply (`true` or `false`) in any concrete decision problem in our applied mechanics applications and this seems to be really significant and it constitutes a step towards artificial intelligence in elementary applied mechanics problems. Although it would be really interesting to have efficient computer implementations of all of the available methods for theorem proving (see, e.g., Fitting (1996)) and have them available for our applied mechanics applications in order to correctly decide in such an application and, more generally, proceed to quantifier eliminations automatically in the symbolic environment, this seems not presently feasible and, therefore, Collins' down-to-the-earth and conceptually simple semi-numerical CAD algorithm (using simple cells/clusters and concrete representative points) is probably the best related possibility and, if it is efficiently and systematically used in applied mechanics problems in the future, it may offer a really new and advantageous avenue towards the derivation of significant new symbolic/logical results in applied mechanics research.

As far as the present author is concerned, he hopes to become able, in the future, to really use `qepcad` inside SACLIB for the efficient solution of CQE problems in applied mechanics with the aid of CAD. In the author's personal opinion, the `qepcad` package, in spite of its minor weak points, constitutes an efficient tool for the researcher in applied mechanics.

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